

OPTIMAL INVESTMENT AND CONSUMPTION WITH LIQUID AND ILLIQUID ASSETS

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ABSTRACT. We consider an optimal investment/consumption problem to maximize expected utility from consumption. In this market model, the investor is allowed to choose a portfolio which consists of one bond, one liquid risky asset (no transaction costs) and one illiquid risky asset (proportional transaction costs). We fully characterize the optimal trading and consumption strategies in terms of the solution of the free boundary ODE with an integral constraint. In the analysis, there is no technical assumption (except a natural one) on the model parameters. We also provide an asymptotic expansion result for small transaction costs.

1. INTRODUCTION

In the seminar papers [25, 26], Merton formulated and solved the optimal investment and consumption problem in the continuous-time stochastic control framework. Under the assumption that the risky asset price process is a geometric Brownian motion and the investor has a CRRA (constant relative risk aversion) utility function, Merton proved that it is optimal to invest a constant proportion of wealth in the risky asset. Since then, the dynamic optimal investment/consumption problems have been studied by many researchers, and the results extend to very general situations (e.g., [18, 19, 22, 21, 15]), under the simplifying assumption of no transaction costs (perfect liquidation).

One type of generalization of these problems is to consider transaction costs which are levied on each transaction. Constantinides and Magill [24] assumed proportional transaction costs in the model of [26]. They intuited that the optimal strategy is to keep the proportion of wealth invested in the risky asset in an interval, by trading the risky asset in a minimal way. Davis and Norman [9] proved this intuition by formulating the HJB (Hamilton-Jacobi-Bellman) equation. Shreve and Soner [29] subsequently complemented the analysis of [9], by removing various technical conditions and using the technique of viscosity solutions to clarify the key arguments. Since the solution of the HJB equation is not explicit, except the case of no transaction costs case, the asymptotic analysis for small transaction costs has been also studied (for a single risky asset case, e.g., see [29, 16, 2, 11, 6]).

The market model in Davis and Norman [9] and Shreve and Soner [29] consists of a single risky asset. Even though the natural extension is to consider a model with multiple risky assets, it is known that transaction costs models with multiple assets are notably harder to analyze than a model with a single risky asset. Consequently, most of the existing results are limited to models with a single risky asset.

For the multiple-asset models, Akian *et al.* [1] prove that the value function is the viscosity solution of the variational inequality. Liu [23] considers the model with exponential utility and independent Brownian motions: In this special case, the multiple-asset problem can be decomposed into a set of the single risky asset problems. Muthuraman and Kumar [27] develop a numerical method to solve the multiple-asset problem. Chen and Dai [5] characterize the shape of the no-trading region in the model with two risky assets. Bichuch and Shreve [4] prove an asymptotic expansion for small transaction costs, in the market with two futures. Possamai *et al.* [28] prove

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an asymptotic expansion for small transaction costs for general Markovian risky asset processes. Because fully rigorous characterization of the optimal strategies is unknown in the models with multiple assets ([23] is an exception), these papers [1, 27, 5, 4, 28] focus on asymptotic analysis or some characteristics of the no-trading region.

In this paper, we consider an optimal investment/consumption problem in the market which consists of one bond, one liquid risky asset and one illiquid risky asset. The investor need to pay proportional transaction costs for trading the illiquid asset, but the other risky asset is perfectly liquid. For CRRA utility functions and infinite time horizon, we fully characterize the value function and the optimal trading/consumption strategies in terms of the solution of a free boundary ODE. Our analysis do not rely on technical assumptions for the market parameters, e.g., the size of transaction costs,¹ or the location of Merton line.² Especially, our main theorem is valid for any size of transaction costs, and the result can be applied to a market model with illiquid asset with high transaction costs. The only assumption we use for the market parameter is a natural one, which is related to the finiteness of the value function. We also prove an asymptotic expansion result for small transaction costs.

Our model is similar to the models in [8, 3, 14]. Dai *et al* [8] consider a model with a finite horizon and position constraints, and they characterize the trading boundaries. Guasoni and Bichuch [3] consider the problem of maximizing the long-term growth rate. Under the assumption of small transaction costs, they solve the problem using the *shadow price approach*, and prove an asymptotic expansion result. In parallel with our work, Hobson *et al.* [14] recently consider a similar problem as in this paper and solve the problem by studying the HJB equation of the primal optimization problem.

In this paper, we employ the *shadow price approach* used in [17, 11, 7, 10, 6, 13, 3]. The shadow price approach amounts to construct the most unfavorable frictionless market, where the asset price processes lie between the bid and ask prices of the original market. After proving that the constructed frictionless market produces the same expected utility as the original market, we obtain the expressions of the optimal strategies and value function by solving the optimization problem in the frictionless market. We find a candidate of the shadow price process using the solution of a constraint free boundary ODE, and do the verification. Eventually, our analysis does not rely on the dynamic programming principle or the technique of viscosity solutions.

The remainder of the paper is organized as follows: Section 2 describes the model. In Section 3, we explain shadow price approach, and heuristically derive a free boundary ODE from the property of the shadow price process. In Section 4, we state and prove the main results: The expression of the optimal strategy and the value function, and the asymptotic expansion for small transaction costs. Finally, Section 5 is devoted to prove the existence of a smooth solution to the free boundary ODE with an integral constraint.

2. THE MODEL

The market model we consider consists of one zero-interest bond³ and two risky assets, whose price processes $S^{(1)}$ and $S^{(2)}$ are given by

$$dS^{(i)} = S^{(i)}(\mu_i dt + \sigma_i dB_t^{(i)}), \quad S_0^{(i)} > 0, \quad i = 1, 2. \quad (2.1)$$

Here, $B^{(1)}$ and $B^{(2)}$ are standard Brownian motions with correlation $\rho \in (-1, 1)$, and the parameters μ_i and σ_i are positive constants. The information structure is given by the augmented filtration generated by $B^{(1)}$ and $B^{(2)}$. We assume that $S^{(2)}$ can be traded without transaction costs, but

¹In [10, 11, 28, 13, 3], the transaction costs term is assumed to be small enough.

²In [17, 11], the Location of Merton line is assume to be in the first quadrant.

³The case with non-zero constant interest rate can be transformed to the case with zero interest rate.

proportional transaction costs are imposed whenever an investor trades $S^{(1)}$. We call $S^{(1)}$ an illiquid asset and $S^{(2)}$ a liquid asset. To be specific, there are constants $\bar{\lambda} > 0$ and $\underline{\lambda} \in (0, 1)$ such that the investor pays $\bar{S}_t^{(1)} := (1 + \bar{\lambda})S_t^{(1)}$ for one share of the illiquid asset, but only gets $\underline{S}_t^{(1)} := (1 - \underline{\lambda})S_t^{(1)}$ for one share of the illiquid asset.

Let the investor initially hold η_0 shares of the bond, η_1 shares of illiquid asset, and η_2 shares of liquid asset. As notation, let the triple $(\varphi_t^{(0)}, \varphi_t^{(1)}, \varphi_t^{(2)})$ represents the number of shares in the bond and two risky assets at time t , and let c_t be the consumption rate. In order to incorporate the possibility of the initial jump, we distinguish $(\varphi_{0-}^{(0)}, \varphi_{0-}^{(1)}, \varphi_{0-}^{(2)})$ and $(\varphi_0^{(0)}, \varphi_0^{(1)}, \varphi_0^{(2)})$. The processes are right-continuous after that. We set $(\varphi_{0-}^{(0)}, \varphi_{0-}^{(1)}, \varphi_{0-}^{(2)}) = (\eta_0, \eta_1, \eta_2)$.

Definition 2.1. \mathcal{C} is a set of nonnegative, right-continuous, and locally integrable optional processes, such that $c \in \mathcal{C}$ if there exist right-continuous optional processes $(\varphi^{(0)}, \varphi^{(1)}, \varphi^{(2)})$ which satisfy the following three conditions:

- (i) $\varphi^{(1)}$ is of finite variation a.s.
- (ii) (Admissibility) Liquidated value is always nonnegative, i.e.,

$$\varphi_t^{(0)} + \underline{S}_t^{(1)}(\varphi_t^{(1)})^+ - \bar{S}_t^{(1)}(\varphi_t^{(1)})^- + S_t^{(2)}\varphi_t^{(2)} \geq 0, \quad t \geq 0. \quad (2.2)$$

- (iii) (Budget constraint) The consumption stream is financeable, i.e.,

$$\varphi_t^{(0)} + \varphi_t^{(2)} S_t^{(2)} = \eta_0 + \eta_2 S_0^{(2)} + \int_0^t \varphi_u^{(2)} dS_u^{(2)} - \int_0^t \bar{S}_u^{(1)} d(\varphi_u^{(1)})^\uparrow + \int_0^t \underline{S}_u^{(1)} d(\varphi_u^{(1)})^\downarrow - \int_0^t c_u du, \quad (2.3)$$

where $(\varphi_t^{(1)})^\uparrow$ and $(\varphi_t^{(1)})^\downarrow$ are the cumulative number of illiquid asset bought and sold up to time t .

For the initial admissibility, we assume that

$$\eta_0 + \underline{S}_t^{(1)}(\eta_1)^+ - \bar{S}_t^{(1)}(\eta_1)^- + S_t^{(2)}\eta_2 \geq 0.$$

For $p \in (-\infty, 1) \setminus \{0\}$, we consider the **utility function** $U : [0, \infty) \rightarrow [-\infty, \infty)$ of the power (CRRA) type. It is defined for $c \geq 0$ by

$$U(c) = \frac{c^p}{p}, \text{ and } U(0) = \begin{cases} 0, & p > 0, \\ -\infty, & p \leq 0 \end{cases}$$

Our goal is to analyze the optimal investment and consumption problem:

$$\sup_{c \in \mathcal{C}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right], \quad (2.4)$$

where the constant $\delta > 0$ is the impatience rate.

Remark 2.2. If there is no transaction costs, i.e., $\bar{\lambda} = \underline{\lambda} = 0$, then the HJB equation has an explicit solution (see Theorem 2.1 in [9]). From the explicit solution, it is easily derived that the value of the optimization problem is finite if and only if

$$\delta > \frac{q}{2(1-\rho^2)} \left(\left(\frac{\mu_1}{\sigma_1} \right)^2 + \left(\frac{\mu_2}{\sigma_2} \right)^2 - 2\rho \frac{\mu_1 \mu_2}{\sigma_1 \sigma_2} \right),$$

where $q := p/(1-p)$.

Remark 2.3. The special cases $\mu_1 = \frac{\rho \mu_2 \sigma_1}{\sigma_2}$ or $\mu_2 = \frac{\rho \sigma_1 \sigma_2}{1+q}$ are covered by the result of [7] regarding the single risky asset case:

- (1) Assume that $\mu_1 = \frac{\rho \mu_2 \sigma_1}{\sigma_2}$. If there is no transaction costs, then it is optimal to hold 0 shares of $S^{(1)}$. This implies that the optimal strategy in the original transaction costs model never trade the asset $S^{(1)}$, and the problem is reduced to the frictionless model with $S^{(2)}$ only.
- (2) Assume that $\mu_2 = \frac{\rho \sigma_1 \sigma_2}{1+q}$. One can check that the problem reduces to the single risky asset problem in [7] with the parameters $\mu^* = \mu_1 - \frac{\rho^2 \sigma_1^2}{(1+q)}$, $\sigma^* = \sigma_1 \sqrt{1 - \rho^2}$, $\delta^* = \delta - \frac{q \rho^2 \sigma_1^2}{2(1+q)^2}$.

Based on Remark 2.2 and Remark 2.3, we impose the following assumption throughout the rest of the paper.

Assumption 2.4. The parameters of the optimization problem satisfy the following conditions:

$$\delta > \frac{q}{2(1-\rho^2)} \left(\left(\frac{\mu_1}{\sigma_1} \right)^2 + \left(\frac{\mu_2}{\sigma_2} \right)^2 - 2\rho \frac{\mu_1 \mu_2}{\sigma_1 \sigma_2} \right), \quad \mu_1 \neq \frac{\rho \mu_2 \sigma_1}{\sigma_2} \quad \text{and} \quad \mu_2 \neq \frac{\rho \sigma_1 \sigma_2}{1+q}.$$

3. HEURISTICS WITH SHADOW PRICE PROCESS

In this section, we explain so called shadow price approach in this context, and heuristically derive a free boundary ODE from the desired property of the shadow price process.

3.1. Shadow price approach. In the shadow price approach (see [17, 11, 7, 10, 6, 3]), the original transaction cost problem is solved by constructing a suitable frictionless (i.e., no transaction costs) market model. We first define the set of consistent price processes, and a set of financeable consumptions in the frictionless market, in Definition 3.1. Then the definition of the shadow price process is given in Definition 3.3.

Definition 3.1. (1) The set of consistent price processes \mathcal{S} is defined as

$$\mathcal{S} = \left\{ \tilde{S} : \tilde{S} \text{ is an Ito-process, and } \underline{S}_t^{(1)} \leq \tilde{S}_t \leq \bar{S}_t^{(1)} \text{ for all } t \geq 0, \text{ a.s.} \right\} \quad (3.1)$$

(2) For each $\tilde{S} \in \mathcal{S}$, $\mathcal{C}(\tilde{S})$ is a set of financeable consumptions in the frictionless market with risky assets \tilde{S} and $S^{(2)}$. To be specific, the set $\mathcal{C}(\tilde{S})$ is defined as a set of nonnegative, locally integrable progressively measurable processes c , such that $c \in \mathcal{C}(\tilde{S})$ if there exist progressively measurable processes $(\varphi^{(0)}, \varphi^{(1)}, \varphi^{(2)})$ which satisfy the following two conditions:

(i) (Admissibility) Total wealth (W for notation) is always nonnegative, i.e.,

$$W_t := \varphi_t^{(0)} + \tilde{S}_t \varphi_t^{(1)} + S_t^{(2)} \varphi_t^{(2)} \geq 0, \quad t \geq 0. \quad (3.2)$$

(ii) (Budget constraint) The consumption stream is financeable, i.e.,

$$W_t = W_{0-} + \int_0^t \varphi_u^{(1)} d\tilde{S}_t + \int_0^t \varphi_u^{(2)} dS_u^{(2)} - \int_0^t c_u du, \quad t \geq 0. \quad (3.3)$$

The connection between the original transaction cost problem and the collection of frictionless problems is described in the following proposition. It is a simple translation of Proposition 2.2 in [7].

Proposition 3.2. *The following two statements hold.*

(1) For each $\tilde{S} \in \mathcal{S}$,

$$\sup_{c \in \mathcal{C}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right] \leq \sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right]. \quad (3.4)$$

(2) Given $\tilde{S} \in \mathcal{S}$, let $\hat{c} \in \mathcal{C}(\tilde{S})$ solve the frictionless optimization problem, i.e.,

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} U(\hat{c}_t) dt \right] = \sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right], \quad (3.5)$$

with $(\hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)})$ which satisfies the budget constraint (3.3). Assume that

(i) $\hat{\varphi}^{(1)}$ is a right-continuous process of finite variation,

(ii) $(\hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)})$ satisfies (2.2),

(iii) $d(\hat{\varphi}_t^{(1)})^\uparrow = 1_{\{\tilde{S}_t = \bar{S}_t\}} d(\hat{\varphi}_t^{(1)})^\uparrow$ and $d(\hat{\varphi}_t^{(1)})^\downarrow = 1_{\{\tilde{S}_t = \underline{S}_t\}} d(\hat{\varphi}_t^{(1)})^\downarrow$.

(iv) $\hat{c}, \hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)}$ are continuous processes except a possible initial jump at $t = 0-$.

Then $\hat{c} \in \mathcal{C}$, and \hat{c} solves the original optimization problem (2.4), i.e.,

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} U(\hat{c}_t) dt \right] = \sup_{c \in \mathcal{C}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right]. \quad (3.6)$$

Proof. (1) For any $c \in \mathcal{C}$, there exists $(\varphi^{(0)}, \varphi^{(1)}, \varphi^{(2)})$ which satisfies (2.3).

$$\begin{aligned} \varphi_t^{(0)} + \varphi_t^{(2)} S_t^{(2)} &= \eta_0 + \eta_2 S_0^{(2)} + \int_0^t \varphi_u^{(2)} dS_u^{(2)} - \int_0^t \bar{S}_u^{(1)} d(\varphi_u^{(1)})^\uparrow + \int_0^t \underline{S}_u^{(1)} d(\varphi_u^{(1)})^\downarrow - \int_0^t c_u du \\ &\leq \eta_0 + \eta_2 S_0^{(2)} + \int_0^t \varphi_u^{(2)} dS_u^{(2)} - \int_0^t \tilde{S}_u d\varphi_u^{(1)} - \int_0^t c_u du, \end{aligned}$$

where the inequality is due to $\tilde{S} \in \mathcal{S}$. Then the integration-by-parts formula produces

$$\varphi_t^{(0)} + \varphi_t^{(1)} \tilde{S}_t + \varphi_t^{(2)} S_t^{(2)} \leq \eta_0 + \eta_1 \tilde{S}_0 + \eta_2 S_0^{(2)} + \int_0^t \varphi_u^{(1)} d\tilde{S}_u + \int_0^t \varphi_u^{(2)} dS_u^{(2)} - \int_0^t c_u du.$$

Therefore, if we define $\tilde{\varphi}^{(0)}$ as

$$\tilde{\varphi}_t^{(0)} := \eta_0 + \eta_1 \tilde{S}_0 + \eta_2 S_0^{(2)} + \int_0^t \varphi_u^{(1)} d\tilde{S}_u + \int_0^t \varphi_u^{(2)} dS_u^{(2)} - \int_0^t c_u du - \varphi_t^{(1)} \tilde{S}_t - \varphi_t^{(2)} S_t^{(2)},$$

then $\tilde{\varphi}^{(0)} \geq \varphi^{(0)}$ and (3.3) is satisfied with $(\tilde{\varphi}^{(0)}, \varphi^{(1)}, \varphi^{(2)}, c)$. We also check (3.2),

$$0 \leq \varphi_t^{(0)} + \underline{S}_t^{(1)} (\varphi_t^{(1)})^+ - \bar{S}_t^{(1)} (\varphi_t^{(1)})^- + S_t^{(2)} \varphi_t^{(2)} \leq \tilde{\varphi}_t^{(0)} + \varphi_t^{(1)} \tilde{S}_t + \varphi_t^{(2)} S_t^{(2)}.$$

Therefore, $c \in \mathcal{C}(\tilde{S})$ and the inclusion $\mathcal{C} \in \mathcal{C}(\tilde{S})$ finishes the proof of (1).

(2) Let $(\hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)}, \hat{c})$ satisfies the assumptions in the proposition. Then by (3.3) and the integration-by-parts formula,

$$\begin{aligned} \hat{\varphi}_t^{(0)} + \hat{\varphi}_t^{(2)} S_t^{(2)} &= -\hat{\varphi}_t^{(1)} \tilde{S}_t + \eta_0 + \eta_1 \tilde{S}_0 + \eta_2 S_0^{(2)} + \int_0^t \hat{\varphi}_u^{(1)} d\tilde{S}_u + \int_0^t \hat{\varphi}_u^{(2)} dS_u^{(2)} - \int_0^t \hat{c}_u du \\ &= \eta_0 + \eta_2 S_0^{(2)} - \int_0^t \tilde{S}_u d\hat{\varphi}_u^{(1)} + \int_0^t \hat{\varphi}_u^{(2)} dS_u^{(2)} - \int_0^t \hat{c}_u du \\ &= \eta_0 + \eta_2 S_0^{(2)} - \int_0^t \bar{S}_u d(\hat{\varphi}_u^{(1)})^\uparrow + \int_0^t \underline{S}_u d(\hat{\varphi}_u^{(1)})^\downarrow + \int_0^t \hat{\varphi}_u^{(2)} dS_u^{(2)} - \int_0^t \hat{c}_u du \end{aligned}$$

Hence (2.3) is satisfied, and $\hat{c} \in \mathcal{C}$. Then (3.4) and (3.5) imply (3.6). \square

Definition 3.3. If $\tilde{S} \in \mathcal{S}$ satisfies following equality, then \tilde{S} is called a shadow price process:

$$\sup_{c \in \mathcal{C}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right] = \sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right]. \quad (3.7)$$

Proposition 3.2 (2) implies that we can solve the original transaction costs problem by solving the frictionless problem with shadow price process, and Proposition 3.2 (1) says that the shadow price process can be characterized as the solution of the following minimization problem:

$$\inf_{\tilde{S} \in \mathcal{S}} \left(\sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right] \right). \quad (3.8)$$

3.2. Heuristic derivation of the free boundary ODE. For the rest of this section, we will heuristically derive a free boundary ordinary differential equation, from the HJB equation for the optimization problem (3.8). For $\tilde{S} \in \mathcal{S}$, we express $\tilde{S}_t = S_t^{(1)} e^{Y_t}$ for an Ito-process Y . Since $1 - \underline{\lambda} \leq \tilde{S}_t / S_t^{(1)} \leq 1 + \bar{\lambda}$, we have a natural bound $Y_t \in [\underline{y}, \bar{y}]$, where $\underline{y} := \ln(1 - \underline{\lambda})$ and $\bar{y} := \ln(1 + \bar{\lambda})$. Assume that the dynamics of Y is given by

$$dY_t = m_t dt + s_{1t} dB_t^{(1)} + s_{2t} dB_t^{(2)}, \quad (3.9)$$

for some processes m, s_1, s_2 . Then the state price density process H , in the market with stock prices \tilde{S} and $S^{(2)}$, satisfies the stochastic differential equation

$$dH_t = -H_t \left(\theta_1(m_t, s_{1t}, s_{2t}) dB_t^{(1)} + \theta_2(m_t, s_{1t}, s_{2t}) dB_t^{(2)} \right), \quad H_0 = 1, \quad (3.10)$$

where the functions θ_1 and θ_2 are defined as

$$\begin{aligned}\theta_1(m, s_1, s_2) &:= \frac{\rho(\sigma_2 s_2 - \mu_2)}{(1-\rho^2)\sigma_2} - \frac{\mu_2 s_2 - (m + \mu_1 + s_1 \sigma_1 + \frac{1}{2}(s_1^2 + s_2^2))\sigma_2}{(1-\rho^2)\sigma_2(s_1 + \sigma_1)}, \\ \theta_2(m, s_1, s_2) &:= \frac{\mu_2}{\sigma_2} - \rho \theta_1(m, s_1, s_2).\end{aligned}\quad (3.11)$$

Since the frictionless market model with stock prices \tilde{S} and $S^{(2)}$ is complete, the standard duality theory can be applied (i.e., see [20]):

$$\begin{aligned}& \sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right] \\ &= \inf_{z > 0} \left(\sup_c \left(\mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right] + z \left((\eta_0 + \tilde{S}_0 \eta_1 + S_0^{(2)} \eta_2) - \mathbb{E} \left[\int_0^\infty c_t H_t dt \right] \right) \right) \right) \\ &= \frac{(\eta_0 + \tilde{S}_0 \eta_1 + S_0^{(2)} \eta_2)^p}{p} \left(\mathbb{E} \left[\int_0^\infty e^{-(1+q)\delta t} H_t^{-q} dt \right] \right)^{1-p},\end{aligned}\quad (3.12)$$

where $q = p/(1-p)$. Consequently, we may rewrite (3.8) as

$$\inf_{\tilde{S} \in \mathcal{S}} \left(\sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right] \right) = \inf_{Y_0} \left\{ \frac{(\eta_0 + S_0^{(1)} e^{Y_0} \eta_1 + S_0^{(2)} \eta_2)^p}{p} |w(Y_0)|^{1-p} \right\}, \quad (3.13)$$

with

$$w(y) := \inf_{m, s_1, s_2} \left\{ \text{sgn}(p) \mathbb{E} \left[\int_0^\infty e^{-(1+q)\delta t} H_t^{-q} dt \mid Y_0 = y \right] \right\}. \quad (3.14)$$

The formal HJB equation for (3.14) has the following form:

$$\inf_{m, s_1, s_2} \left\{ -\alpha(m, s_1, s_2) w(y) + (m + \beta(m, s_1, s_2)) w'(y) + \gamma(s_1, s_2) w''(y) + \text{sgn}(p) \right\} = 0, \quad (3.15)$$

where (with $\theta_1 = \theta_1(m, s_1, s_2)$ and $\theta_2 = \theta_2(m, s_1, s_2)$)

$$\begin{aligned}\alpha(m, s_1, s_2) &:= (1+q)\delta - \frac{q(1+q)}{2} (\theta_1^2 + \theta_2^2 + 2\rho \theta_1 \theta_2), \\ \beta(m, s_1, s_2) &:= q((s_1 + \rho s_2)\theta_1 + (\rho s_1 + s_2)\theta_2), \\ \gamma(s_1, s_2) &:= \frac{1}{2}(s_1^2 + s_2^2 + 2\rho s_1 s_2).\end{aligned}\quad (3.16)$$

To incorporate the requirement $Y_t \in [y, \bar{y}]$, we turn off the diffusion ($s_{1t} = s_{2t} = 0$) whenever Y_t reaches the boundary \underline{y} or \bar{y} , and let the drift be the inward direction. By observing the form of the minimizer in (3.15), we infer that the boundary condition would be

$$w''(\underline{y}) = w''(\bar{y}) = \infty. \quad (3.17)$$

To handle this infinite boundary condition and reduce the order of the differential equation, we change variable. Let $x = -w'(y)$ and define the function $g : [\underline{x}, \bar{x}] \mapsto \mathbb{R}$ as $g(x) = w(y)$, with $\underline{x} = -w'(\bar{y})$ and $\bar{x} = -w'(\underline{y})$. With x and g , (3.15) is written as

$$\inf_{m, s_1, s_2} \left\{ -\alpha(m, s_1, s_2) g(x) - (m + \beta(m, s_1, s_2)) x + \gamma(s_1, s_2) \frac{x}{g'(x)} + \text{sgn}(p) \right\} = 0, \quad x \in [\underline{x}, \bar{x}]. \quad (3.18)$$

(3.17) and the relation $dy/dx = -g'(x)/x$ produce a boundary condition and an integral constraint:

$$g'(\underline{x}) = g'(\bar{x}) = 0, \quad \int_{\underline{x}}^{\bar{x}} \frac{g'(x)}{x} dx = \bar{y} - \underline{y}. \quad (3.19)$$

Since \underline{x} and \bar{x} are not predetermined, (3.18) together with (3.19) is a free boundary problem with an integral constraint.

Remark 3.4. The purpose of this section is only to derive the free boundary problem which we analyze rigorously in the next section: The arguments in this section is heuristic and not rigorous.

4. THE RESULTS

In this section, we first present the existence result for the solution of the free boundary problem that we derived in the previous section. Then we construct the candidate shadow price process \tilde{S} using the solution of the free boundary problem. In Lemma 4.4, we solve the optimization problem for the market with the candidate shadow price process. In Theorem 4.5, we verify that the \tilde{S} is indeed the shadow price process by checking the conditions in Proposition 3.2 (2), and conclude that the optimal solution in Lemma 4.4 also solves the original transaction cost problem (2.4). Finally, asymptotic expansion of the no-trading region for small transaction costs is given in Corollary 4.8.

The proofs of results related to the free boundary problem are postponed to Section 5 due to their technical nature.

Proposition 4.1. *Under Assumption 2.4, there exist constants \underline{x}, \bar{x} and a function $g \in C^2[\underline{x}, \bar{x}]$ that satisfy following conditions:*

- (1) *If $\mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}$, then $0 < \underline{x} < \bar{x}$. If $\mu_1 < \frac{\rho\mu_2\sigma_1}{\sigma_2}$, then $\underline{x} < \bar{x} < 0$.*
- (2) *For $x \in [\underline{x}, \bar{x}]$, g satisfies the differential equation*

$$\inf_{m, s_1, s_2} \left\{ -\alpha(m, s_1, s_2)g(x) - (m + \beta(m, s_1, s_2))x + \gamma(s_1, s_2)\frac{x}{g'(x)} + \text{sgn}(p) \right\} = 0, \quad (4.1)$$

where the functions α, β, γ are given in (3.11) and (3.16).

- (3) *The following boundary/integral conditions are satisfied:*

$$g'(\underline{x}) = g'(\bar{x}) = 0 \text{ and } \int_{\underline{x}}^{\bar{x}} \frac{g'(x)}{x} dx = \log\left(\frac{1+\bar{\lambda}}{1-\bar{\lambda}}\right). \quad (4.2)$$

- (4) *The functions*

$$qg(x), \quad qg(x)(g'(x) + 1) - (1 + q)xg'(x), \quad q(g(x) - xg'(x)), \quad \text{and} \quad g'(x) + 1$$

are strictly positive on $[\underline{x}, \bar{x}]$. Recall that $q = p/(1 - p)$.

- (5) *$g'(x)/x > 0$ for $x \in (\underline{x}, \bar{x})$.*

Proof. See Section 5. □

We need the next corollary to construct the shadow price process.

Corollary 4.2. (1) *The minimizer $(\hat{m}(x), \hat{s}_1(x), \hat{s}_2(x))$ of (4.1) is well defined on $[\underline{x}, \bar{x}]$.*

(2) *Let $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\theta}_1, \hat{\theta}_2 : [\underline{x}, \bar{x}] \mapsto \mathbb{R}$ be the composition of the functions $\alpha, \beta, \gamma, \theta_1, \theta_2$ of (3.16) and (3.11) with the optimizers $\hat{m}, \hat{s}_1, \hat{s}_2$ of (4.1). For instance, $\hat{\alpha}(x) := \alpha(\hat{m}(x), \hat{s}_1(x), \hat{s}_2(x))$. Then the following functions are Lipschitz on $[\underline{x}, \bar{x}]$:*

$$\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\theta}_1, \hat{\theta}_2, \frac{\hat{s}_1(x)}{g'(x)}, \frac{\hat{s}_2(x)}{g'(x)}, \frac{\hat{\beta}(x)}{g'(x)}. \quad (4.3)$$

- (3) *For $x \in [\underline{x}, \bar{x}]$, we have*

$$-\hat{\alpha}(x)g'(x) - (\hat{m}(x) + \hat{\beta}(x)) + \hat{\gamma}(x)\left(\frac{x}{g'(x)}\right)' = 0. \quad (4.4)$$

Proof. See Section 5. □

We construct the shadow price process using the solution $(g, \bar{x}, \underline{x})$ of the free boundary problem in Proposition 4.1. As a preliminary, we define the functions $f, \xi, r : [\underline{x}, \bar{x}] \mapsto \mathbb{R}$ as

$$\begin{aligned} f(x) &:= \underline{y} + \int_x^{\bar{x}} \frac{g'(t)}{t} dt, \\ \xi(x) &:= \eta_0 + \eta_1 S_0^{(1)} e^{f(x)} + \eta_2 S_0^{(2)}, \\ r(x) &:= \eta_1 S_0^{(1)} e^{f(x)} - \xi(x) \frac{x}{qg(x)}, \end{aligned} \quad (4.5)$$

where $\underline{y} = \ln(1 - \underline{\lambda})$ and $\bar{y} = \ln(1 + \bar{\lambda})$. Then $e^{f(\underline{x})} = (1 + \bar{\lambda})$ and $e^{f(\bar{x})} = (1 - \underline{\lambda})$. Let the constant $\hat{x} \in [\underline{x}, \bar{x}]$ be defined by

$$\hat{x} = \begin{cases} \bar{x}, & r(x) > 0 \text{ for all } x \in [\underline{x}, \bar{x}] \\ \underline{x}, & r(x) < 0 \text{ for all } x \in [\underline{x}, \bar{x}] \\ \text{a solution to } r(x) = 0, & \text{otherwise.} \end{cases} \quad (4.6)$$

Consider the following reflected (Skorokhod-type) SDE on the interval $[\underline{x}, \bar{x}]$:

$$\begin{cases} dX_t = \left(X_t \hat{\alpha}(X_t) + \frac{X_t \hat{\beta}(X_t)}{g'(X_t)} \right) dt - \frac{X_t \hat{s}_1(X_t)}{g'(X_t)} dB_t^{(1)} - \frac{X_t \hat{s}_2(X_t)}{g'(X_t)} dB_t^{(2)} + d\Phi_t \\ X_0 = \hat{x}. \end{cases} \quad (4.7)$$

Corollary 4.2 (2) implies that the coefficients of the above SDE are Lipschitz on $[\underline{x}, \bar{x}]$. Therefore, the classical result of [30] is applicable: (4.7) has a unique solution (X, Φ) such that Φ is a continuous process of finite variation and satisfies

$$d\Phi_t^\uparrow = 1_{\{X_t = \underline{x}\}} d\Phi_t^\uparrow, \quad d\Phi_t^\downarrow = 1_{\{X_t = \bar{x}\}} d\Phi_t^\downarrow. \quad (4.8)$$

We define the process (candidate shadow price process) \tilde{S} as

$$\tilde{S}_t := S_t^{(1)} e^{f(X_t)} \quad (4.9)$$

The intuition is following: In Section 3, we change variable (y, w) to (x, g) , and they satisfy $dy/dx = -g'(x)/x$ and $-w'(\underline{y}) = \bar{x}$, which implies $y = f(x)$. Also in Section 3, the shadow price process has the form of $S_t^{(1)} e^{Y_t}$.

Proposition 4.3. (1) $\underline{S}_t^{(1)} \leq \tilde{S}_t \leq \bar{S}_t^{(1)}$ for $t \geq 0$ a.s.

(2) \tilde{S}_t satisfies the SDE

$$\begin{aligned} \frac{d\tilde{S}_t}{\tilde{S}_t} &= \left(\hat{m}(X_t) + \mu_1 + \sigma_1(\hat{s}_1(X_t) + \rho \hat{s}_2(X_t)) + \hat{\gamma}(X_t) \right) dt \\ &\quad + (\hat{s}_1(X_t) + \sigma_1) dB_t^{(1)} + \hat{s}_2(X_t) dB_t^{(2)} \end{aligned} \quad (4.10)$$

Proof. (1) Proposition 4.1 (iv) implies that f is a monotonically decreasing function. Hence $\underline{y} \leq f(x) \leq \bar{y}$, which implies $\underline{S}_t^{(1)} \leq \tilde{S}_t \leq \bar{S}_t^{(1)}$.

(2) By Ito's formula,

$$\begin{aligned} d(f(X_t)) &= \left(-\frac{g'(x)}{x} \left(x \hat{\alpha}(x) + \frac{x \hat{\beta}(x)}{g'(x)} \right) + \left(\frac{x}{g'(x)} \right)' \hat{\gamma}(x) \right) \Big|_{x=X_t} dt \\ &\quad + \hat{s}_1(X_t) dB_t^{(1)} + \hat{s}_2(X_t) dB_t^{(2)} - \frac{g'(x)}{x} d\Phi_t \\ &= \hat{m}(X_t) dt + \hat{s}_1(X_t) dB_t^{(1)} + \hat{s}_2(X_t) dB_t^{(2)}, \end{aligned} \quad (4.11)$$

where the dt term is simplified by (4.4), and the reflection term $(\frac{g'(x)}{x} d\Phi_t)$ vanishes because of $g'(\underline{x}) = g'(\bar{x}) = 0$ and (4.8). Ito's formula for $\tilde{S}_t = S_t^{(1)} e^{f(X_t)}$, together with (2.1) and (4.11), produces (4.10). \square

In the frictionless market with $(\tilde{S}, S^{(2)})$, the state price density process \hat{H} is given by

$$\frac{d\hat{H}_t}{\hat{H}_t} = -\hat{\theta}_1(X_t) dB_t^{(1)} - \hat{\theta}_2(X_t) dB_t^{(2)}, \quad \hat{H}_0 = 1. \quad (4.12)$$

Consider the optimization problem in the frictionless market $(\tilde{S}, S^{(2)})$:

$$\sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right]. \quad (4.13)$$

In the next lemma, we characterize the value and the optimal strategy for (4.13).

Lemma 4.4. (1) Let \tilde{S} and \hat{H} be as in (4.9) and (4.12). Then

$$\sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right] = \frac{\xi(\hat{x})^p}{p} |g(\hat{x})|^{1-p}. \quad (4.14)$$

(2) In (4.14), the optimal wealth \hat{W} and the optimal investment/consumption $(\hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)}, \hat{c})$ can be written as following:

$$\begin{aligned} \hat{W}_t &= \xi(\hat{x}) e^{-(1+q)\delta t} \hat{H}_t^{-(1+q)} \frac{g(X_t)}{g(\hat{x})}, \\ \hat{\varphi}_t^{(0)} &= (1 - \pi_1(X_t) - \pi_2(X_t)) \hat{W}_t, \quad \hat{\varphi}_t^{(1)} = \frac{\pi_1(X_t) \hat{W}_t}{\hat{S}_t}, \quad \hat{\varphi}_t^{(2)} = \frac{\pi_2(X_t) \hat{W}_t}{S_t^{(2)}}, \quad \hat{c}_t = \frac{\hat{W}_t}{|g(X_t)|}, \end{aligned} \quad (4.15)$$

where the functions $\pi_1, \pi_2 : [\underline{x}, \bar{x}] \mapsto \mathbb{R}$ are

$$\begin{aligned} \pi_1(x) &:= \frac{(1+q)\hat{\theta}_1(x) - \frac{x\hat{s}_1(x)}{g(x)}}{\hat{s}_1(x) + \sigma_1}, \\ \pi_2(x) &:= \frac{1}{\sigma_2} \left((1+q)\hat{\theta}_2(x)g(x) - \frac{x\hat{s}_2(x)}{g(x)} - \pi_1(x)\hat{s}_2(x) \right). \end{aligned} \quad (4.16)$$

Proof. (1) The standard duality theory for complete market model (see, e.g., Theorem 9.11, p. 141 in [20]) implies that

$$\sup_{c \in \mathcal{C}(\tilde{S})} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right] = \frac{\xi(\hat{x})^p}{p} \left(\mathbb{E} \left[\int_0^\infty e^{-(1+q)\delta t} \hat{H}_t^{-q} dt \right] \right)^{1-p}, \quad (4.17)$$

and the optimal consumption \hat{c} is

$$\hat{c}_t = \frac{\xi(\hat{x}) e^{-(1+q)\delta t} \hat{H}_t^{-(1+q)}}{\mathbb{E} \left[\int_0^\infty e^{-(1+q)\delta t} \hat{H}_t^{-q} dt \right]}. \quad (4.18)$$

To prove (4.14), it is enough to show that

$$g(\hat{x}) = \text{sgn}(p) \mathbb{E} \left[\int_0^\infty e^{-(1+q)\delta t} \hat{H}_t^{-q} dt \right]. \quad (4.19)$$

Using Ito formula and (4.4), we have

$$d(g(X_t)) = \left(-X_t \hat{m}(X_t) + \frac{X_t \hat{\gamma}(X_t)}{g'(X_t)} \right) dt - X_t \hat{s}_1(X_t) dB_t^{(1)} - X_t \hat{s}_2(X_t) dB_t^{(2)}, \quad (4.20)$$

where the reflection term vanishes because of $g'(\underline{x}) = g'(\bar{x}) = 0$ and (4.8).

Observe that the stochastic exponential $\mathcal{E}(q\hat{\theta} \cdot B)$, with $\hat{\theta}_t = (\hat{\theta}_1(X_t), \hat{\theta}_2(X_t))$ and $B_t = (B_t^{(1)}, B_t^{(2)})$, is a martingale since $\hat{\theta}$ is bounded. Let $\bar{B}^{(1)}, \bar{B}^{(2)}$ be defined by

$$\bar{B}_t^{(1)} := B_t^{(1)} - q \int_0^t \hat{\theta}_1(X_s) + \rho \hat{\theta}_2(X_s) ds, \quad \bar{B}_t^{(2)} := B_t^{(2)} - q \int_0^t \rho \hat{\theta}_1(X_s) + \hat{\theta}_2(X_s) ds.$$

Since $\hat{\theta}_1$ and $\hat{\theta}_2$ are bounded on $[\underline{x}, \bar{x}]$, by Girsanov's theorem, $\bar{B}^{(1)}$ and $\bar{B}^{(2)}$ are Brownian motions on $[0, t]$ under the measure $\bar{\mathbb{P}}_t$, defined by $d\bar{\mathbb{P}}_t = \mathcal{E}(q\hat{\theta} \cdot B)_t d\mathbb{P}$. Then,

$$\begin{aligned} & \mathbb{E}^{\bar{\mathbb{P}}_t} \left[e^{-\int_0^t \hat{\alpha}(X_u) du} g(X_t) \right] \\ &= g(\hat{x}) - \mathbb{E}^{\bar{\mathbb{P}}_t} \left[\int_0^t e^{-\int_0^u \hat{\alpha}(X_s) ds} \left(\text{sgn}(p) + X_u (\hat{s}_1(X_u) d\bar{B}_u^{(1)} + \hat{s}_2(X_u) d\bar{B}_u^{(2)}) \right) du \right] \\ &= g(\hat{x}) - \text{sgn}(p) \mathbb{E}^{\bar{\mathbb{P}}_t} \left[\int_0^t e^{-\int_0^u \hat{\alpha}(X_s) ds} du \right] \\ &= g(\hat{x}) - \text{sgn}(p) \mathbb{E} \left[\int_0^t e^{-(1+q)\delta u} \hat{H}_u^{-q} du \right] \end{aligned} \quad (4.21)$$

Here the first equality uses Ito formula and (4.1), and the second equality holds because $B_t^{(1)}$ and $B_t^{(2)}$ are Brownian motions under the measure $\bar{\mathbb{P}}_t$ and the integrands are bounded. The third equality is due to (4.12) and $d\bar{\mathbb{P}}_t = \mathcal{E}(q\hat{\theta} \cdot B)_t d\mathbb{P}$.

We have two cases to consider, $p > 0$ and $p < 0$.

(i) In case $p > 0$: Since $g(x)$ is positive (see Proposition 4.1 (iv)), (4.21) implies that

$$\mathbb{E}\left[\int_0^\infty e^{-(1+q)\delta t} \hat{H}_t^{-q} dt\right] < \infty.$$

Hence, there exists a sequence $(t_n)_{n \in \mathbb{N}}$ with $t_n \rightarrow \infty$ such that $\mathbb{E}[e^{-(1+q)\delta t_n} \hat{H}_{t_n}^{-q}] \rightarrow 0$. Since g is bounded, we also have $\mathbb{E}[e^{-(1+q)\delta t_n} \hat{H}_{t_n}^{-q} g(X_{t_n})] \rightarrow 0$. After taking limit in (4.21) with t_n , we conclude (4.19).

(ii) In case $p < 0$: From the form of the function α in (3.16) and $q < 0$, we have $\hat{\alpha} > (1+q)\delta$. Since g is bounded,

$$\left| \mathbb{E}^{\bar{\mathbb{P}}_t} \left[e^{-\int_0^t \hat{\alpha}(X_u) du} g(X_t) \right] \right| \leq |g|_\infty e^{-(1+q)\delta t} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (4.22)$$

Let $t \rightarrow \infty$ in (4.21), we conclude (4.19).

(2) Obviously, $\hat{W}_t > 0$ for $t \geq 0$. Since we have (4.18) and (4.19), it is enough to check the budget constraint in Definition 3.1 (2). It can be written as

$$\frac{d\hat{W}_t}{\hat{W}_t} = \pi_1(X_t) \frac{d\tilde{S}_t}{\tilde{S}_t} + \pi_2(X_t) \frac{dS_t^{(2)}}{S_t^{(2)}} - \frac{\hat{c}_t}{\hat{W}_t} dt. \quad (4.23)$$

Using Ito formula with (4.12), (4.20), (4.10), (4.16) and (4.1), one can check that the budget constraint holds (the computation is rather long and tedious but elementary, so it is omitted). \square

Now we are ready to state our main result. In Theorem 4.5, we verify that the process \tilde{S} in (4.9) is indeed a shadow price process. Consequently, the optimal trading/consumption strategy $(\hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)}, \hat{c})$ of the frictionless problem (4.13) also satisfies (2.2) and (2.3), and $\hat{c} \in \mathcal{C}$ is the optimizer of (2.4).

Theorem 4.5. *The processes $(\hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)}, \hat{c})$ in (4.15) solves (2.4). In other words, $(\hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)}, \hat{c})$ satisfies the conditions in Definition 2.1 (therefore $\hat{c} \in \mathcal{C}$), and*

$$\sup_{c \in \mathcal{C}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(c_t) dt \right] = \mathbb{E} \left[\int_0^\infty e^{-\delta t} U(\hat{c}_t) dt \right] = \frac{\xi(\hat{x})^p}{p} |g(\hat{x})|^{1-p}.$$

Proof. By Lemma 4.4, we already know that $(\hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)}, \hat{c})$ is the optimal solution of (3.5). Therefore, we only need to check that $(\hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)}, \hat{c})$ satisfies the assumptions in Proposition 3.2 (2). Then, the result of Proposition 3.2 completes the proof of this theorem.

Let's first consider the initial jump. We need to show that the assumption (iii) in Proposition 3.2 (2) is satisfied at $t = 0$, which can be written as

$$\hat{\varphi}_0^{(1)} - \eta_1 = 1_{\{\hat{x}=\underline{x}\}}(\hat{\varphi}_0^{(1)} - \eta_1)^+ - 1_{\{\hat{x}=\bar{x}\}}(\hat{\varphi}_0^{(1)} - \eta_1)^-. \quad (4.24)$$

In (4.16), we can simplify $\pi_1(x)$ as $\pi_1(x) = \frac{x}{qg(x)}$ by using expressions in (5.1). Then $r(x)$ in (4.5) can be written as $r(x) = \eta_1 e^{f(x)} S_0^{(1)} - \xi(x) \pi_1(x)$. Now we can see why we defined $\hat{x} \in [\underline{x}, \bar{x}]$ as (4.6). The three possibilities are described below:

$$\begin{cases} \hat{\varphi}_0^{(1)} = \eta_1, & \text{if } r(\hat{x}) = 0, \\ \hat{\varphi}_0^{(1)} < \eta_1 \text{ and } \hat{x} = \bar{x}, & \text{if } r(x) > 0 \text{ on } [\underline{x}, \bar{x}], \\ \hat{\varphi}_0^{(1)} > \eta_1 \text{ and } \hat{x} = \underline{x}, & \text{if } r(x) < 0 \text{ on } [\underline{x}, \bar{x}]. \end{cases} \quad (4.25)$$

Obviously (4.25) implies (4.24), and we conclude that the assumption (iii) in Proposition 3.2 (2) is satisfied at $t = 0$.

By Proposition 4.1 and the form of π_1 , we observe that $\hat{\varphi}_t^{(1)} > 0$ if $\mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}$ and $\hat{\varphi}_t^{(1)} < 0$ if $\mu_1 < \frac{\rho\mu_2\sigma_1}{\sigma_2}$. With (4.10), (4.7), (4.23) and (4.20), Ito formula produces (after a long but straightforward computation)

$$\begin{cases} d(\ln(\hat{\varphi}_t^{(1)})) = d\left(\ln\left(\frac{\pi_1(X_t)\hat{W}_t}{\hat{S}_t}\right)\right) = \frac{1}{X_t}d\Phi_t, & \text{when } \mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}, \\ d(\ln(-\hat{\varphi}_t^{(1)})) = d\left(\ln\left(\frac{-\pi_1(X_t)\hat{W}_t}{\hat{S}_t}\right)\right) = \frac{1}{X_t}d\Phi_t, & \text{when } \mu_1 < \frac{\rho\mu_2\sigma_1}{\sigma_2}. \end{cases} \quad (4.26)$$

(4.26) and (4.8) implies that the assumptions (i) and (iii) in Proposition 3.2 (2) are satisfied.

Since the assumption (iv) is obvious, it remains to check the assumption (ii) in Proposition 3.2 (2). This amounts to prove that

$$\hat{\varphi}_t^{(0)} + \underline{S}_t^{(1)}(\hat{\varphi}_t^{(1)})^+ - \bar{S}_t^{(1)}(\hat{\varphi}_t^{(1)})^- + S_t^{(2)}\hat{\varphi}_t^{(2)} \geq 0, \quad t \geq 0. \quad (4.27)$$

Using Proposition 4.1 (4) and (5), we obtain following inequalities:

$$\begin{aligned} \frac{d}{dx}\left(\frac{\pi_1(x)e^{-f(x)}}{1-\pi_1(x)}\right) &= \frac{(qg(x)(g'(x)+1)-(1+q)xg'(x))e^{-f(x)}}{q^2g(x)^2(1-\pi_1(x))^2} > 0, \quad x \in [\underline{x}, \bar{x}] \\ \frac{d}{dx}\pi_1(x) &= \frac{q(g(x)-xg'(x))}{q^2g(x)^2} > 0, \quad x \in [\underline{x}, \bar{x}] \end{aligned} \quad (4.28)$$

• In case $\mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}$: By Proposition 4.1 (1) and (3), we have $\pi_1(x) > 0$, so $\hat{\varphi}_t^{(1)} > 0$.

If $\pi_1(X_t) \leq 1$, then $\hat{\varphi}_t^{(0)} + \hat{\varphi}_t^{(2)}S_t^{(2)} = (1 - \pi_1(X_t))\hat{W}_t \geq 0$. Therefore, (4.27) holds.

If $\pi_1(X_t) > 1$, then $\hat{\varphi}_t^{(0)} + \hat{\varphi}_t^{(2)}S_t^{(2)} < 0$. (4.28) implies that

$$\frac{\hat{\varphi}_t^{(0)} + \underline{S}_t^{(1)}\hat{\varphi}_t^{(1)} + S_t^{(2)}\hat{\varphi}_t^{(2)}}{\hat{\varphi}_t^{(0)} + S_t^{(2)}\hat{\varphi}_t^{(2)}} = (1 - \lambda)\frac{\pi_1(X_t)e^{-f(X_t)}}{1-\pi_1(X_t)} + 1 \leq (1 - \lambda)\frac{\pi_1(\bar{x})e^{-f(\bar{x})}}{1-\pi_1(\bar{x})} + 1 = \frac{1}{1-\pi_1(\bar{x})} < 0,$$

where we use $e^{-f(\bar{x})} = 1/(1 - \lambda)$. Hence (4.27) holds.

• In case $\mu_1 < \frac{\rho\mu_2\sigma_1}{\sigma_2}$: By Proposition 4.1 (1) and (3), we have $\pi_1(x) < 0$, so $\hat{\varphi}_t^{(1)} < 0$ and $\hat{\varphi}_t^{(0)} + \hat{\varphi}_t^{(2)}S_t^{(2)} = (1 - \pi_1(X_t))\hat{W}_t > 0$. (4.28) implies that

$$\frac{\hat{\varphi}_t^{(0)} + \bar{S}_t^{(1)}\hat{\varphi}_t^{(1)} + S_t^{(2)}\hat{\varphi}_t^{(2)}}{\hat{\varphi}_t^{(0)} + S_t^{(2)}\hat{\varphi}_t^{(2)}} = (1 + \bar{\lambda})\frac{\pi_1(X_t)e^{-f(X_t)}}{1-\pi_1(X_t)} + 1 \geq (1 + \bar{\lambda})\frac{\pi_1(\underline{x})e^{-f(\underline{x})}}{1-\pi_1(\underline{x})} + 1 = \frac{1}{1-\pi_1(\underline{x})} > 0,$$

where we use $e^{-f(\underline{x})} = 1/(1 + \bar{\lambda})$. Hence (4.27) holds.

We showed that $(\hat{\varphi}^{(0)}, \hat{\varphi}^{(1)}, \hat{\varphi}^{(2)}, \hat{c})$ satisfies the assumptions in Proposition 3.2 (2), and the proof is completed by the result of Proposition 3.2. \square

We have more explicit characterization for the optimal investment in the illiquid asset.

Corollary 4.6. *In (2.4), it is optimal to minimally trade the illiquid asset $S^{(1)}$ in such a way that the proportion of investment in illiquid asset is within the interval $[\underline{\pi}, \bar{\pi}]$, i.e.,*

$$\underline{\pi} \leq \frac{\hat{\varphi}_t^{(1)}S_t^{(1)}}{\hat{\varphi}_t^{(0)} + \hat{\varphi}_t^{(1)}S_t^{(1)} + \hat{\varphi}_t^{(2)}S_t^{(2)}} \leq \bar{\pi}, \quad (4.29)$$

where $\underline{\pi}, \bar{\pi} \in \mathbb{R}$ have explicit expressions in terms of $g, \underline{x}, \bar{x}$ in Proposition 4.1:

$$\underline{\pi} := \frac{\pi_1(\underline{x})}{\pi_1(\underline{x}) + (1 + \bar{\lambda})(1 - \pi_1(\underline{x}))}, \quad \bar{\pi} := \frac{\pi_1(\bar{x})}{\pi_1(\bar{x}) + (1 - \lambda)(1 - \pi_1(\bar{x}))} \quad (4.30)$$

Proof. We can easily transform

$$\pi_1(X_t) = \frac{\hat{\varphi}_t^{(1)}\tilde{S}_t}{\hat{\varphi}_t^{(0)} + \hat{\varphi}_t^{(1)}\tilde{S}_t + \hat{\varphi}_t^{(2)}S_t^{(2)}} \implies \frac{\hat{\varphi}_t^{(1)}S_t^{(1)}}{\hat{\varphi}_t^{(0)} + \hat{\varphi}_t^{(1)}S_t^{(1)} + \hat{\varphi}_t^{(2)}S_t^{(2)}} = \frac{\pi_1(X_t)}{\pi_1(X_t) + (1 - \pi_1(X_t))e^{f(X_t)}}.$$

Direct computation produces following inequality:

$$\frac{d}{dx}\left(\frac{\pi_1(x)}{\pi_1(x) + (1 - \pi_1(x))e^{f(x)}}\right) = \frac{(qg(x)(g'(x)+1)-(1+q)xg'(x))e^{f(x)}}{q^2g(x)^2((e^{f(x)}-1)\pi_1(x)-e^{f(x)})^2} > 0, \quad x \in [\underline{x}, \bar{x}],$$

where we use the result in Proposition 4.1 (4). Therefore, we have

$$\frac{\pi_1(\underline{x})}{\pi_1(\underline{x}) + (1+\lambda)(1-\pi_1(\underline{x}))} \leq \frac{\pi_1(X_t)}{\pi_1(X_t) + (1-\pi_1(X_t))e^{f(X_t)}} \leq \frac{\pi_1(\bar{x})}{\pi_1(\bar{x}) + (1-\lambda)(1-\pi_1(\bar{x}))}, \quad t \geq 0,$$

and the result follows. \square

Remark 4.7. As we pointed out in Remark 2.3, we can describe results similar to Theorem 4.5 and Corollary 4.6 for the case of $\mu_1 = \frac{\rho\mu_2\sigma_1}{\sigma_2}$ or $\mu_2 = \frac{\rho\sigma_1\sigma_2}{1+q}$, by using the results in [7].

We also compute the asymptotic expansions of the no-trading region $\bar{\pi}$ and $\underline{\pi}$ for small transaction costs. In fact, we can prove that $\bar{\pi}$ and $\underline{\pi}$ can be written as power series expansions of $\lambda^{\frac{1}{3}}$, as in [6].

Corollary 4.8. *For simplicity, let $\bar{\lambda} = 0$ and $\underline{\lambda} = \lambda$. Then $\bar{\pi}$ and $\underline{\pi}$ are analytic functions of $\lambda^{\frac{1}{3}}$, for small enough $\lambda > 0$. We can recursively compute the coefficients of their power series expansions. The first two terms are*

$$\underline{\pi} = \zeta_0 - \zeta_1 \lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}) \quad \text{and} \quad \bar{\pi} = \zeta_0 + \zeta_1 \lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}), \quad (4.31)$$

where

$$\begin{aligned} \zeta_0 &= \frac{(1+q)(\mu_1 - \frac{\rho\sigma_1\mu_2}{\sigma_2})}{(1-\rho^2)}, \\ \zeta_1 &= \left(\frac{3(1+q)^3(\mu_1 - \frac{\rho\sigma_1\mu_2}{\sigma_2})^2 \left((1+q)^2\sigma_1^2\mu_2^2 - 2\rho(1+q)^2\sigma_1\sigma_2\mu_1\mu_2 + ((1+q)^2\mu_1^2 + (2(1+q)\mu_1 - \sigma_1^2)(\rho^2 - 1)\sigma_1^2)\sigma_2^2 \right)}{4(1-\rho^2)^4\sigma_1^8\sigma_2^2} \right)^{\frac{1}{3}}. \end{aligned} \quad (4.32)$$

Proof. We can prove the result by following the proof in [6] in a parallel manner, so we omit the details. \square

Remark 4.9. When $\rho = 0$, we observe that ζ_1 is simplified as

$$\zeta_1 = \left(\frac{3\mu_1^2\mu_2^2(1+q)^5}{4\sigma_1^6\sigma_2^2} + \frac{3\mu_1^2(1+q)^3((1+q)\mu_1 - \sigma_1^2)^2}{4\sigma_1^8} \right)^{\frac{1}{3}}.$$

In [6], the model with one illiquid asset (without $S^{(2)}$) is considered, and the coefficient of $\lambda^{\frac{1}{3}}$ for the expansion for the no-trading region is given by

$$\left(\frac{3\mu_1^2(1+q)^3((1+q)\mu_1 - \sigma_1^2)^2}{4\sigma_1^8} \right)^{\frac{1}{3}} \lambda^{\frac{1}{3}}.$$

Therefore, when the Brownian motions are uncorrelated, the additional investment opportunity for the liquid risky asset makes the no-trading region wider (for small enough transaction costs). This phenomenon is also observed in [3].

5. PROOFS

We start with the simpler one, Corollary 4.2, by using Proposition 4.1. After that, we prove Proposition 4.1.

Proof. (Proof of Corollary 4.2)

(1) (4.1) is a simple optimization of a quadratic function. By Proposition 4.1 (4) and (5), we can see that the first order condition produces the minimizer as follows:

$$\begin{aligned} \hat{s}_1(x) &= \frac{\sigma_1(x - qg(x))g'(x)}{qg(x)(1+g'(x)) - (1+q)xg'(x)}, \\ \hat{s}_2(x) &= \frac{(-\rho\sigma_1\sigma_2x - (\mu_2(1+q)^2 - q\rho\sigma_1\sigma_2)xg'(x) + q(1+q)\mu_2g(x)(1+g'(x)))g'(x)}{\sigma_2(qg(x)(1+g'(x)) - (1+q)xg'(x))(1+g'(x))}, \\ \hat{m}(x) &= \frac{1}{2q(1+q)\sigma_2g(x)} \left(2(1-\rho^2)\sigma_2(\sigma_1 + (1+q)\hat{s}_1(x))(\sigma_1\hat{s}_1(x))x \right. \\ &\quad + q(1+q) \left(2\mu_2(\rho(\sigma_1 + \hat{s}_1(x)) + \hat{s}_2(x)) \right. \\ &\quad \left. \left. - \sigma_2(2\mu_1 + 2\sigma_1\hat{s}_1(x) + \hat{s}_1(x)^2 + 2\rho(\sigma_1 + \hat{s}_1(x))\hat{s}_2(x) + \hat{s}_2(x)^2) \right) g(x) \right). \end{aligned} \quad (5.1)$$

Therefore, $(\hat{m}(x), \hat{s}_1(x), \hat{s}_2(x))$ are well defined on $[\underline{x}, \bar{x}]$ because of Proposition 4.1 (4).

(2) The form of $\hat{s}_1(x)$ and $\hat{s}_2(x)$ above, and the observation

$$\hat{s}_1(x) + \sigma_1 = \frac{q\sigma_1(g(x)-xg'(x))}{qg(x)(1+g'(x))-(1+q)xg'(x)} \neq 0 \quad \text{for } x \in [\underline{x}, \bar{x}],$$

show that the functions in (4.3) are Lipschitz on $[\underline{x}, \bar{x}]$, by Proposition 4.1 (4).

(3) The appropriate version of the Envelope Theorem (see, e.g., Theorem 3.3, p. 475 in [12]) or the direct computation produces (4.4).

□

The rest of this section is devoted to the proof of Proposition 4.1. We first define notation for convenience, and prove Lemma 5.1. Then Proposition 5.2, Proposition 5.3, Proposition 5.4 and Proposition 5.5 deal with four different sub-cases and complete the proof of Proposition 4.1.

Using the expression (5.1) of optimizers, we rewrite (4.1) as

$$\frac{A(x, g(x))g'(x)^2 + B(x, g(x))g'(x) + C(x, g(x))}{2(1+q)\sigma_2^2(1+g'(x))(qg(x)(1+g'(x))-(1+q)xg'(x))} = 0, \quad (5.2)$$

where

$$\begin{aligned} A(x, y) &= -2(1+q)^2\sigma_2^2 \operatorname{sgn}(p)x \\ &\quad + ((1+q)^4\mu_2^2 - 2\rho q(1+q)^2\sigma_1\sigma_2\mu_2 + ((1+2q+q^2\rho^2)\sigma_1^2 - 2(1+q)^2\mu_1)\sigma_2^2)x^2 \\ &\quad + (1+q)\left(2q\sigma_2^2 \operatorname{sgn}(p) + (2\rho q^2\sigma_1\sigma_2\mu_2 - 2q(1+q)^2\mu_2^2 + (2\delta(1+q)^2 + q(2\mu_1 - \sigma_1^2))\sigma_2^2)x\right)y \\ &\quad - q(1+q)^2(2\delta\sigma_2^2 - q\mu_2^2)y^2 \\ B(x, y) &= -2(1+q)^2\sigma_2^2 \operatorname{sgn}(p)x + 2\sigma_2(\rho\sigma_1\mu_2(1+q)^2 - (\mu_1(1+q)^2 - q(1-\rho^2)\sigma_1^2)\sigma_2)x^2 \\ &\quad + (1+q)\left(4q\sigma_2^2 \operatorname{sgn}(p) + (2\rho q(q-1)\sigma_1\sigma_2\mu_2 - 2q(1+q)^2\mu_2^2 + (2\delta(1+q)^2 + q(4\mu_1 - \sigma_1^2))\sigma_2^2)x\right)y \\ &\quad - 2q(1+q)^2(2\delta\sigma_2^2 - q\mu_2^2)y^2 \\ C(x, y) &= -(1-\rho^2)\sigma_1^2\sigma_2^2x^2 + 2q(1+q)\sigma_2(\operatorname{sgn}(p)\sigma_2 + (\mu_1\sigma_2 - \rho\mu_2\sigma_1)x)y \\ &\quad - q(1+q)^2(2\delta\sigma_2^2 - q\mu_2^2)y^2 \end{aligned} \quad (5.3)$$

As notation, let Δ_x be a discriminant of quadratic equation with respect to x , i.e., $\Delta_x(ax^2 + bx + c) = b^2 - 4ac$.

y_C , x_D , y_D , x_M and y_M are defined as

$$\begin{aligned} y_C &:= \frac{2\sigma_2^2 \operatorname{sgn}(p)}{(1+q)(2\delta\sigma_2^2 - q\mu_2^2)}, \\ x_D &:= \frac{2q(1+q) \operatorname{sgn}(p)}{2\delta(1+q)^2 + q(\sigma_1^2 - 2(1+q)\mu_1)}, \\ y_D &:= \frac{2(1+q) \operatorname{sgn}(p)}{2\delta(1+q)^2 + q(\sigma_1^2 - 2(1+q)\mu_1)}, \\ x_M &:= \frac{q(\mu_1 - \frac{\rho\sigma_1\mu_2}{\sigma_2}) \operatorname{sgn}(p)}{(1-\rho^2)\sigma_1^2(\delta - \frac{q}{2(1-\rho^2)}((\frac{\mu_1}{\sigma_1})^2 + (\frac{\mu_1}{\sigma_2})^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2}))}, \\ y_M &:= \frac{\operatorname{sgn}(p)}{\delta - \frac{q}{2(1-\rho^2)}((\frac{\mu_1}{\sigma_1})^2 + (\frac{\mu_1}{\sigma_2})^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2})}. \end{aligned} \quad (5.4)$$

Lemma 5.1. *Let A, B, C are functions as in (5.3). Then the following statements hold:*

- (1) $\{(x, y) : B(x, y) = C(x, y) = 0\} = \{(x, y) : B(x, y) = A(x, y) = 0\} = \{(0, 0), (0, y_C)\}$.
- (2) $\{(x, y) : x \neq 0, B(x, y)^2 - 4A(x, y)C(x, y) = 0\} = \{(x_D, y_D)\}$.
- (3) $\{(x, y) : x \neq 0, B(x, y)^2 - 4A(x, y)C(x, y) < 0\} = \emptyset$.

Proof. (1) Suppose that $(x, y) \in \mathbb{R}^2$ satisfies $B(x, y) = C(x, y) = 0$. Then

$$\begin{aligned} 0 &= 2C(x, y) - B(x, y) \\ &= (1+q)x \left((2q(1+q)^2\mu_2^2 - 2\rho q(1+q)\sigma_1\sigma_2\mu_2 + (q\sigma_1^2 - 2\delta(1+q)^2)\sigma_2^2)y \right. \\ &\quad \left. + 2\sigma_2((1+q)\sigma_2 \operatorname{sgn}(p) + (\sigma_2((1+q)\mu_1 - (1-\rho^2)\sigma_1^2) - \rho(1+q)\sigma_1\mu_2)x) \right) \end{aligned}$$

There are three possibilities:

(i) In case $2q(1+q)^2\mu_2^2 - 2\rho q(1+q)\sigma_1\sigma_2\mu_2 + (q\sigma_1^2 - 2\delta(1+q)^2)\sigma_2^2 = 0$ and $(1+q)\sigma_2 \operatorname{sgn}(p) + (\sigma_2((1+q)\mu_1 - (1-\rho^2)\sigma_1^2) - \rho(1+q)\sigma_1\mu_2)x = 0$, we solve these equations for δ and x and obtain

$$\delta = \frac{q(2(1+q)^2\mu_2^2 - 2\rho(1+q)\sigma_1\sigma_2\mu_2 + \sigma_1^2\sigma_2^2)}{2(1+q)^2\sigma_2^2} \text{ and } x = \frac{(1+q)\sigma_2 \operatorname{sgn}(p)}{\rho(1+q)\sigma_1\mu_2 - (\mu_1(1+q) - (1-\rho^2)\sigma_1^2)\sigma_2}.$$

Substitute these expressions for δ and x , we obtain

$$\Delta_y(C(x, y)) = -(1-\rho^2) \left(\frac{2q(1+q)\sigma_1\sigma_2^2((1+q)\mu_2 - \rho\sigma_1\sigma_2)}{\rho(1+q)\sigma_1\mu_2 - (\mu_1(1+q) - (1-\rho^2)\sigma_1^2)\sigma_2} \right)^2 < 0.$$

Therefore, $C(x, y) \neq 0$, which is a contradiction.

(ii) In case $y = -\frac{2\sigma_2((1+q)\sigma_2 \operatorname{sgn}(p) + (\sigma_2((1+q)\mu_1 - (1-\rho^2)\sigma_1^2) - \rho(1+q)\sigma_1\mu_2)x)}{2q(1+q)^2\mu_2^2 - 2\rho q(1+q)\sigma_1\sigma_2\mu_2 + (q\sigma_1^2 - 2\delta(1+q)^2)\sigma_2^2}$, we substitute this expression for y and obtain

$$\Delta_x(C(x, y)) = -(1-\rho^2) \left(\frac{4q(1+q)\sigma_1\sigma_2^3((1+q)\mu_2 - \rho\sigma_1\sigma_2)}{2q(1+q)^2\mu_2^2 - 2\rho q(1+q)\sigma_1\sigma_2\mu_2 + (q\sigma_1^2 - 2\delta(1+q)^2)\sigma_2^2} \right)^2 < 0.$$

Therefore, $C(x, y) \neq 0$, which is a contradiction.

(iii) In case $x = 0$, we solve $B(0, y) = C(0, y) = 0$ for y and obtain $y = 0$ or $y = y_C$. Therefore, $\{(x, y) : B(x, y) = C(x, y) = 0\} = \{(0, 0), (0, y_C)\}$.

The proof of $\{(x, y) : B(x, y) = A(x, y) = 0\} = \{(0, 0), (0, y_C)\}$ is similar.

(2) $B(x, y)^2 - 4A(x, y)C(x, y)$ is quadratic in y . If $x \neq 0$ and $x \neq x_D$, then

$$\begin{aligned} \Delta_y(B(x, y)^2 - 4A(x, y)C(x, y)) \\ = -(1-\rho^2) \left(4(1+q)^2\sigma_1\sigma_2^3((1+q)\mu_2 - \rho\sigma_1\sigma_2)x^2(2\delta(1+q)^2 + q(\sigma_1^2 - 2\mu_1(1+q))) \right. \\ \left. (x - x_D) \right)^2 < 0. \end{aligned}$$

Hence $B(x, y)^2 - 4A(x, y)C(x, y)$ can be zero only when $x = 0$ or $x = x_D$. We have

$$\begin{aligned} B(x_D, y)^2 - 4A(x_D, y)C(x_D, y) \\ = \sigma_2^2 x_D^2 \left((1-\rho^2)\sigma_2^2(2\delta(1+q)^2 - q\sigma_1^2)^2 + (2q(1+q)\sigma_1\mu_2 - \rho\sigma_2(2\delta(1+q)^2 + q\sigma_1^2))^2 \right) (y - y_D)^2, \end{aligned}$$

and observe that $2\delta(1+q)^2 - q\sigma_1^2$ and $2q(1+q)\sigma_1\mu_2 - \rho\sigma_2(2\delta(1+q)^2 + q\sigma_1^2)$ cannot be zero at the same time because $\mu_2 \neq \frac{\rho\sigma_1\sigma_2}{1+q}$. Now we conclude that

$$\{(x, y) : x \neq 0, B(x, y)^2 - 4A(x, y)C(x, y) = 0\} = \{(x_D, y_D)\}$$

(3) In the proof of (2), we showed that

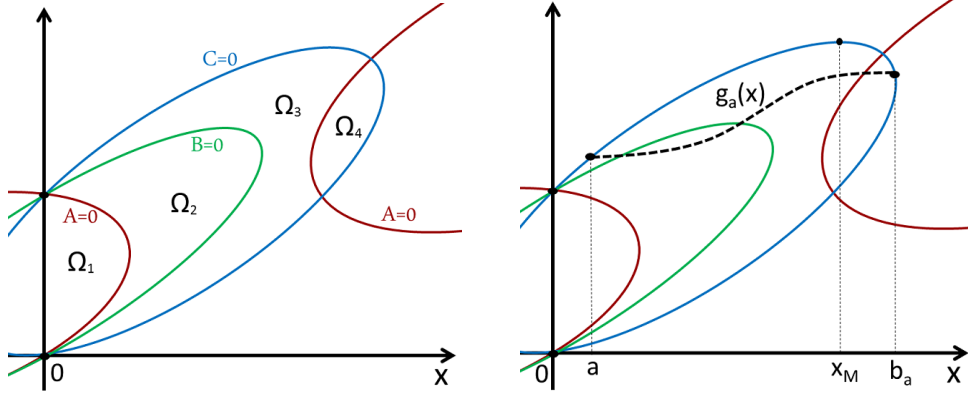
(i) if $x = x_D$, then $B(x, y)^2 - 4A(x, y)C(x, y) \geq 0$,

(ii) if $x \neq 0$ or $x \neq x_D$, then there is no solution to $B(x, y)^2 - 4A(x, y)C(x, y) = 0$.

Therefore, to prove (3), it is enough to show that the coefficient of y^2 in $B(x, y)^2 - 4A(x, y)C(x, y)$ is positive for any $x \neq 0$. Indeed, the coefficient of y^2 in $B(x, y)^2 - 4A(x, y)C(x, y)$ can be written as a sum of squares:

$$((1+q)\sigma_2 x)^2 \left((1-\rho^2)\sigma_2^2(2\delta(1+q)^2 - q\sigma_1^2)^2 + (2q(1+q)\sigma_1\mu_2 - \rho\sigma_2(2\delta(1+q)^2 + q\sigma_1^2))^2 \right) > 0.$$

□

Figure 1. $0 < p < 1$ and $\mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}$

We split the proof of Proposition 4.1 into four propositions which take care of different parameter regimes. In Proposition 5.2, we provide a detailed proof of Proposition 4.1 for the case of $0 < p < 1$ and $\mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}$. Proofs for the other cases are similar, hence we omit some of the details: See Proposition 5.3 for $0 < p < 1$ and $\mu_1 < \frac{\rho\mu_2\sigma_1}{\sigma_2}$; See Proposition 5.4 for $p < 0$ and $\mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}$; See Proposition 5.5 for $p < 0$ and $\mu_1 < \frac{\rho\mu_2\sigma_1}{\sigma_2}$.

Proposition 5.2. *In case $0 < p < 1$ and $\mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}$, Proposition 4.1 holds.*

Proof. We observe that A, B, C are quadratic in x, y . The level curve $C = 0$ is an ellipse, because

$$\frac{\partial^2 C}{\partial x \partial y} - \frac{\partial^2 C}{\partial x^2} \frac{\partial^2 C}{\partial y^2} = -8(1 - \rho^2)q(1 + q)^2\sigma_1^2\sigma_2^4 \left(\delta - \frac{q}{2(1 - \rho^2)} \left(\left(\frac{\mu_1}{\sigma_1} \right)^2 + \left(\frac{\mu_1}{\sigma_2} \right)^2 - 2\rho \frac{\mu_1\mu_2}{\sigma_1\sigma_2} \right) \right) < 0.$$

Cauchy-Schwarz inequality, together with Assumption 2.4, produces

$$2\delta(1 + q)^2 + q(\sigma_1^2 - 2(1 + q)\mu_1) > 0 \text{ and } 2\delta\sigma_2^2 - q\mu_2^2 > 0.$$

Hence y_C, x_D, y_D, x_M, y_M are all positive. The slopes of the level curves $C = 0$, $B = 0$ and $A = 0$ at the points $(0, y_C)$ and $(0, 0)$ are

$$\begin{cases} C = 0 : & \frac{2\sigma_2^2(\mu_1 - \frac{\rho\sigma_1\mu_2}{\sigma_2})}{(1+q)(2\delta\sigma_2^2 - q\mu_2^2)} \text{ and } 0 \\ B = 0 : & \frac{-\mu_2(1+q)^2 - (1-q)\rho\sigma_1\sigma_2\mu_2 + (4\mu_1 - \sigma_1^2)\sigma_2^2}{2(1+q)(2\delta\sigma_2^2 - q\mu_2^2)} \text{ and } \frac{1+q}{2q} \\ A = 0 : & \frac{-\mu_2^2(1+q)^2 + 2\rho q\sigma_1\sigma_2\mu_2 + (2\mu_1 - \sigma_1^2)\sigma_2^2}{(1+q)(2\delta\sigma_2^2 - q\mu_2^2)} \text{ and } \frac{1+q}{q} \end{cases}$$

We observe that

$$\begin{aligned} \frac{2\sigma_2^2(\mu_1 - \frac{\rho\sigma_1\mu_2}{\sigma_2})}{(1+q)(2\delta\sigma_2^2 - q\mu_2^2)} - \frac{-\mu_2(1+q)^2 - (1-q)\rho\sigma_1\sigma_2\mu_2 + (4\mu_1 - \sigma_1^2)\sigma_2^2}{2(1+q)(2\delta\sigma_2^2 - q\mu_2^2)} &= \frac{\mu_2^2(1+q)^2 - 2\mu_2(1+q)\rho\sigma_1\sigma_2 + \sigma_1^2\sigma_2^2}{2(1+q)(2\delta\sigma_2^2 - q\mu_2^2)} > 0, \\ \frac{-\mu_2(1+q)^2 - (1-q)\rho\sigma_1\sigma_2\mu_2 + (4\mu_1 - \sigma_1^2)\sigma_2^2}{2(1+q)(2\delta\sigma_2^2 - q\mu_2^2)} - \frac{-\mu_2^2(1+q)^2 + 2\rho q\sigma_1\sigma_2\mu_2 + (2\mu_1 - \sigma_1^2)\sigma_2^2}{(1+q)(2\delta\sigma_2^2 - q\mu_2^2)} &= \frac{\mu_2^2(1+q)^2 - 2\mu_2(1+q)\rho\sigma_1\sigma_2 + \sigma_1^2\sigma_2^2}{2(1+q)(2\delta\sigma_2^2 - q\mu_2^2)} > 0, \end{aligned}$$

where we use the Cauchy-Schwarz inequality. By this observation and Lemma 5.1 (1), the quadratic curves $A = 0, B = 0, C = 0$ are as in Figure 1. Using the observation that the coefficients of y^2 in A, B, C are all negative, we partition the ellipse as

$$\{(x, y) : x > 0, C(x, y) \geq 0\} = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4,$$

$$\text{where } \begin{cases} \Omega_1 := \{(x, y) : x > 0, C > 0, B \geq 0, A \geq 0\} \\ \Omega_2 := \{(x, y) : x > 0, C > 0, B \geq 0, A < 0\} \\ \Omega_3 := \{(x, y) : x > 0, C \geq 0, B < 0, A < 0\} \\ \Omega_4 := \{(x, y) : x > 0, C \geq 0, B < 0, A \geq 0\} \end{cases}$$

By Lemma 5.1 (3), $\sqrt{B(x,y)^2 - 4A(x,y)C(x,y)}$ is well defined if $x \neq 0$. If $(x,y) \in \Omega_2 \cup \Omega_3 \cup \Omega_4$, then $-B(x,y) + \sqrt{B(x,y)^2 - 4A(x,y)C(x,y)} > 0$.

Claim 1: For any $a \in \mathbb{R}$ such that $0 < a < x_M$, there exist a constant $b_a > a$ and a function $g_a : [a, b_a] \mapsto \mathbb{R}$ such that

$$g'_a(x) = F(x, g_a(x)), \quad g_a(a) = \Gamma(a), \quad g'_a(b_a) = 0,$$

$$\text{where } \begin{cases} F(x, y) := \frac{2C(x, y)}{-B(x, y) + \sqrt{B(x, y)^2 - 4A(x, y)C(x, y)}}, \\ \Gamma(x) := \frac{q\sigma_2(\sigma_2 + (\mu_1\sigma_2 - \rho\sigma_1\mu_2)x) + \sigma_2\sqrt{2q\delta(\rho^2 - 1)\sigma_1^2\sigma_2^2x^2 + q^2(\mu_2^2\sigma_1^2x^2 - 2\rho\sigma_1\sigma_2\mu_2x(1 + \mu_1x) + \sigma_2^2(1 + \mu_1x)^2)}}{q(1+q)(2\delta\sigma_2^2 - q\mu_2^2)}. \end{cases}$$

In fact, $y = \Gamma(x)$ is the equation for the upper part of the ellipse $C = 0$.

(Proof of Claim 1): The function F is continuous and nonnegative on $\Omega_2 \cup \Omega_3 \cup \Omega_4$. By the Peano existence theorem, starting from $(a, \Gamma(a))$, we can evolve the above ODE to the right (see Figure 1) until $(x, g_a(x))$ reaches the boundary of $\Omega_2 \cup \Omega_3 \cup \Omega_4$. Indeed, the curve $(x, g_a(x))$ is inside of $\Omega_2 \cup \Omega_3 \cup \Omega_4$ for $x > a$ close enough to a , because $\Gamma'(a) > 0$ and $g'_a(a) = F(a, \Gamma(a)) = 0$. Let the equation of the upper curve of $\partial\Omega_1 \cap \partial\Omega_2$ be $l(x)$. Observe that $(a, g_a(a))$ is above the curve $\partial\Omega_1 \cap \partial\Omega_2$, i.e., $g_a(a) > l(a)$ (see Figure 1). Define $b_a > a$ as

$$b_a := \inf \{x > a : (x, g_a(x)) \in \partial(\Omega_2 \cup \Omega_3 \cup \Omega_4)\}.$$

Since $F \geq 0$ on $\Omega_2 \cup \Omega_3 \cup \Omega_4$, $\lim_{x \uparrow b_a} g_a(x)$ exists. Suppose that $\lim_{x \uparrow b_a} g_a(x) = l(b_a)$. The definition of b_a implies that $g'_a(x) > 0$ for $a < x < b_a$, and we observe that $-B + \sqrt{B^2 - 4AC} = 0$ on $\partial\Omega_1 \cap \partial\Omega_2$. Then, we produce

$$+\infty = \lim_{x \uparrow b_a} F(x, g_a(x)) = \lim_{x \uparrow b_a} g'_a(x) = \lim_{x \uparrow b_a} \frac{g_a(b_a) - g_a(x)}{b_a - x} \leq \lim_{x \uparrow b_a} \frac{l(b_a) - l(x)}{b_a - x} = l'(b_a) \leq l'(0) < \infty,$$

where we use L'Hospital's Rule and concavity of the curve $l(x)$. This is a contradiction, and we conclude that

$$b_a = \inf \{x > a : C(x, g_a(x)) = 0\}.$$

Then $g'_a(b_a) = F(b_a, g_a(b_a)) = 0$ since $C(b_a, g_a(b_a)) = 0$.

(End of the proof of Claim 1).

Claim 2: $g'_a(x) \neq \frac{1}{q}$ for $x \in [a, b_a]$.

(Proof of Claim 2): Suppose that $\{x \in [a, b_a] : g'_a(x) = \frac{1}{q}\} \neq \emptyset$. Then we define x_0 as

$$x_0 := \inf \{x \in [a, b_a] : g'_a(x) = \frac{1}{q}\}.$$

$F(x_0, y) = \frac{1}{q}$ implies $(2qC(x_0, y) + B(x_0, y))^2 = B(x_0, y)^2 - 4A(x_0, y)C(x_0, y)$. Solving this equation for y , we obtain

$$y = \frac{x}{q} \quad \text{or} \quad y = L(x) := \frac{2(1+q)\sigma_2^2 - (\mu_2^2(1+q)^2 + (\sigma_1^2 - 2(1+q)\mu_1)\sigma_2^2)x}{(1+q)^2(2\delta\sigma_2^2 - q\mu_2^2)}. \quad (5.5)$$

Direct computation produces

$$\begin{aligned} x > 0 \text{ and } F(x, \frac{x}{q}) = \frac{1}{q} &\iff 0 < x \leq x_D \\ x > 0 \text{ and } F(x, L(x)) = \frac{1}{q} &\iff 0 < x \leq x_D \end{aligned}$$

Therefore, $x_0 \leq x_D$. And the point $(x_0, g_a(x_0))$ is on one of the two lines in (5.5). We also check that

$$A(x_D, y_D) > 0, B(x_D, y_D) < 0, C(x_D, y_D) > 0 \implies (x_D, y_D) \in \Omega_4. \quad (5.6)$$

(5.6) implies that the line segment $y = L(x)$ connecting $(0, y_0)$ and (x_D, y_D) is inside of the ellipse $C = 0$. Therefore, $(a, g_a(a))$ is above this line segment. If $(x_0, g_a(x_0))$ is on this line segment, then by the definition of x_0 , $g'_a(x_0) \leq L'(x_0)$. This is a contradiction because

$$g'_a(x_0) - L'(x_0) = \frac{1}{q} - \frac{(2(1+q)\mu_1 - \sigma_1^2)\sigma_2^2 - (1+q)^2\mu_2^2}{(1+q)^2(2\delta\sigma_2^2 - q\mu_2^2)} = \frac{2\delta(1+q)^2 + q(\sigma_1^2 - 2(1+q)\mu_1)}{q(1+q)^2(2\delta\sigma_2^2 - q\mu_2^2)} > 0.$$

The line segment $y = \frac{x}{q}$ connecting $(0, 0)$ and (x_D, y_D) is below the line $y = L(x)$. Therefore, $(x_0, g_a(x_0))$ cannot be on this line segment, neither. Now we reach the contradiction. (End of the proof of Claim 2).

Claim 3: In Claim 1, the solution g_a is unique in $x \in [a, b_a]$ and $g_a \in C^2([a, b_a])$

(Proof of Claim 3): Direct computation produces $F(x_D, y_D) = \frac{1}{q}$. Therefore, by Claim 2, $(x, g_a(x)) \neq (x_D, y_D)$ for $x \in [a, b_a]$. Considering Lemma 5.1 (2) and (3), we can include the set $\{(x, g_a(x)) : x \in [a, b_a]\}$ by a compact set in \mathbb{R}^2 where F is uniformly Lipschitz. Then the uniqueness is from PicardLindelf theorem. Since $F \in C^2(\mathbb{R}_+ \times \mathbb{R} \setminus \{(x_D, y_D)\})$, we conclude $g_a \in C^2([a, b_a])$. (End of the proof of Claim 3).

Claim 4: Let $G(a) := \int_a^{b_a} \frac{g'_a(x)}{x} dx$. Then G has the following properties:

- (i) G is continuous on $(0, x_M)$.
- (ii) $\lim_{x \uparrow x_M} G(x) = 0$.
- (iii) $\lim_{x \downarrow 0} G(x) = 0$.

(Proof of Claim 4): (i) Suppose that $g_a(\cdot)$ is tangent to the ellipse $C = 0$ at $x = b_a$. Since $g'_a(b_a) = 0$, the only possibility is $(b_a, g_a(b_a)) = (x_M, y_M)$. Direct computation produces $g''_a(b_a) = \frac{d}{dx} F(x, g_a(x))|_{x=x_M} = 0$, $\Gamma'(x_M) = 0$ and $\Gamma''(x_M) = -\frac{(1-\rho^2)\sigma_1^2}{q(1+q)} < 0$. This observation implies that $g_a(x) > \Gamma(x)$ for $x < x_M$ close enough to x_M , which is a contradiction. Therefore, g_a is not tangent to $C = 0$ at $x = b_a$, and by the implicit function theorem and the continuity of g_a with respect to the initial data (see, e.g., Theorem VI., p 145 in [31]), the map $a \mapsto b_a$ is continuous.

By Claim 2, $0 \leq g'_a < \frac{1}{q}$ on $[a, b_a]$ for any $a \in (0, x_M)$. Since the map $a \mapsto b_a$ is continuous, G is continuous on $a \in (0, x_M)$ by the dominated convergence theorem.

(ii) The ellipse $C = 0$ has the biggest y value at $x = x_M$. Since g_a increases and $g'_a(b_a) = 0$, we have $b_a \geq x_M$ and $\lim_{a \uparrow x_M} b_a = x_M$. Since $|g'_a| \leq \frac{1}{q}$, the dominated convergence theorem produces

$$\lim_{x \uparrow x_M} |G(x)| \leq \lim_{x \uparrow x_M} \int_a^{b_a} \frac{|g'_a(x)|}{x} dx \leq \lim_{x \uparrow x_M} \frac{1}{q} \ln\left(\frac{b_a}{a}\right) = 0.$$

(iii) Let $\Gamma_k : \mathbb{R} \mapsto \mathbb{R}$ for $k \geq 0$ be defined as

$$\Gamma_k(x) := \max\{y : F(x, y) = k\}.$$

We observe that $\Gamma_k(x)$ is well-defined for small enough k and x , and $\Gamma_k(x) < \Gamma(x)$ for $x > 0$ and $k > 0$, in the intersection of their domains. Also, $\Gamma_0(x) = \Gamma(x)$ and $\Gamma'_0(0) = \frac{2\sigma_2(\sigma_2\mu_1 - \rho\sigma_1\mu_2)}{(1+q)(2\delta\sigma_2^2 - q\mu_2^2)} > 0$. Since $\Gamma'_k(x)$ is jointly continuous for small enough x and k , there exists $\epsilon > 0$ such that

$$\Gamma'_\epsilon(x) > 2\epsilon \quad \text{for } x \in [0, \epsilon].$$

We define $h(x) := x + \frac{\Gamma(x) - \Gamma_\epsilon(x)}{\epsilon}$. Since $\Gamma_\epsilon(0) = \Gamma(0)$, we have $\lim_{x \downarrow 0} h(x) = 0$. Hence, there exists $a_\epsilon > 0$ such that $0 < h(x) < \epsilon$ for their $x \leq a_\epsilon$.

Let $a \in (0, a_\epsilon)$ be fixed. Suppose that $g_a(x) > \Gamma_\epsilon(x)$ for $x \in [a, h(a)]$. Then, the definition of the level curve Γ_ϵ implies that $g'_a(x) < \epsilon$ on $[a, h(a)]$, but this is a contradiction:

$$0 < g_a(h(a)) - \Gamma_\epsilon(h(a)) = \int_a^{h(a)} (g'_a(x) - \Gamma'_\epsilon(x)) dx + \Gamma(a) - \Gamma_\epsilon(a) \leq -\epsilon(h(a) - a) + \Gamma(a) - \Gamma_\epsilon(a) = 0.$$

Therefore, g_a intersect Γ_ϵ on $[a, h(a)]$. After g_a intersect Γ_ϵ , g_a is below Γ_ϵ for a while: $g_a(x) < \Gamma_\epsilon(x)$ for $x \in [h(a), \epsilon]$, because $F(x, \Gamma_\epsilon(x)) = \epsilon < \Gamma'_\epsilon(x)$ for $x \in [0, \epsilon]$. This means that $g'_a(x) \geq \epsilon$ for $x \in [h(a), \epsilon]$.

Now we take the limit $a \downarrow 0$ and obtain the result:

$$\liminf_{a \downarrow 0} G(a) \geq \liminf_{a \downarrow 0} \int_{h(a)}^\epsilon \frac{\epsilon}{x} dx = \liminf_{a \downarrow 0} \epsilon \ln \left(\frac{\epsilon}{h(a)} \right) = \infty.$$

(End of the proof of Claim 4).

By Claim 4 and the intermediate value theorem, we can choose $a \in (0, x_M)$ such that

$$\int_a^{b_a} \frac{g'_a(x)}{x} dx = \ln \left(\frac{1+\bar{\lambda}}{1-\bar{\lambda}} \right).$$

We set $\underline{x} = a$, $\bar{x} = b_a$ and $g = g_a$. Then, \underline{x} , \bar{x} and g satisfy followings:

$$0 < \underline{x} < \bar{x}, g \in C^2([\underline{x}, \bar{x}]) \text{ and}$$

$$g'(x) = F(x, g(x)), \quad g'(\underline{x}) = 0, \quad g'(\bar{x}) = 0, \quad \int_{\underline{x}}^{\bar{x}} \frac{g'(x)}{x} dx = \ln \left(\frac{1+\bar{\lambda}}{1-\bar{\lambda}} \right). \quad (5.7)$$

Here we have $g'(\underline{x}) = 0$ because $C(\underline{x}, \Gamma(\underline{x})) = 0$.

Claim 5: g has the following properties:

- (i) $g(x) > 0$ for $x \in (\underline{x}, \bar{x})$ and $g'(x) > 0$ for $x \in [\underline{x}, \bar{x}]$.
- (ii) $g'(x)/x > 0$ for $x \in (\underline{x}, \bar{x})$.
- (iii) $q g(x)(g'(x) + 1) - (1 + q)xg'(x) > 0$ for $x \in [\underline{x}, \bar{x}]$.
- (iv) $g(x) - xg'(x) > 0$ for $x \in [\underline{x}, \bar{x}]$.

(Proof of Claim 5): (i) and (ii) are obvious from the construction.

(iii) Observe that $q g(\underline{x})(g'(\underline{x}) + 1) - (1 + q)\underline{x}g'(\underline{x}) = q g(\underline{x}) > 0$. Suppose that there exists $x_0 \in [\underline{x}, \bar{x}]$ s.t. $q g(x_0)(g'(x_0) + 1) - (1 + q)x_0g'(x_0) = 0$. Then,

$$g'(x_0) = \frac{q g(x_0)}{(1+q)x_0 - q g(x_0)} \implies F(x_0, g(x_0)) = \frac{q g(x_0)}{(1+q)x_0 - q g(x_0)} \implies g(x_0) = \frac{x_0}{q} \implies g'(x_0) = \frac{1}{q},$$

which contradicts to Claim 2.

(iv) Obviously, $g(\underline{x}) - \underline{x}g'(\underline{x}) = g(\underline{x}) > 0$. Suppose that there exists $x_0 \in [\underline{x}, \bar{x}]$ such that $g(x_0) - x_0g'(x_0) = 0$. We set $k = g'(x_0)$ and $g(x_0) = kx_0$. Then, $F(x_0, kx_0) = k$ can be rewritten as

$$\begin{aligned} & \sqrt{B(x_0, kx_0)^2 - 4A(x_0, kx_0)B(x_0, kx_0)} \\ &= - \frac{4k(1+k)(1+q)^3(1-kq)\sigma_2^2 \left(k^2\mu_2^2(1+q)^2 - 2k^2\mu_2(1+q)\rho\sigma_1\sigma_2 + ((1+k)^2 - (1+2k)\rho^2)\sigma_1^2\sigma_2^2 \right)}{\left(k^2\mu_2^2(1+q)^2(kq-1) + 2k\mu_2(1+q)(kq-1)\rho\sigma_1\sigma_2 - (2k(1+k)(1+q)(\delta k(1+q) - \mu_1) + (\rho^2 - 1 + k(k+q+kq-q\rho^2))\sigma_1^2\sigma_2^2 \right)^2} \end{aligned}$$

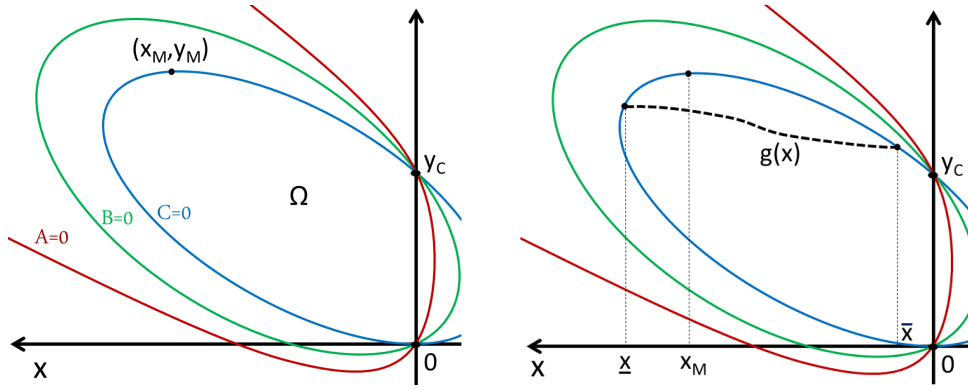
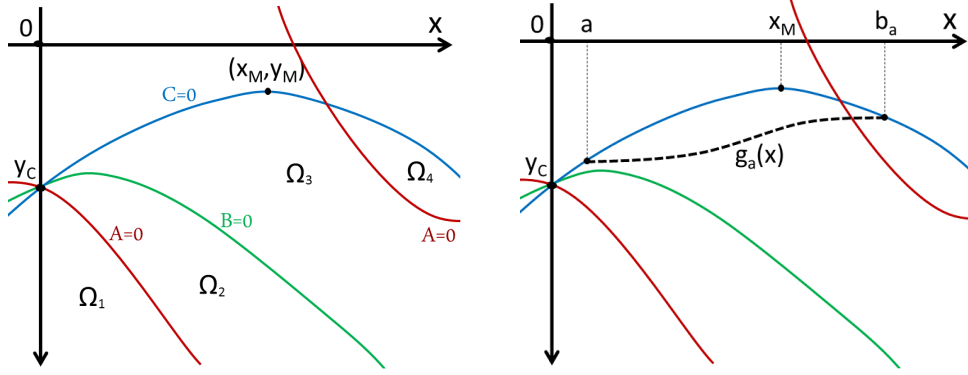
Observe that $1 - kq > 0$ by Claim 2. Then the above equality is a contradiction because

$$\begin{aligned} & \Delta_k(k^2\mu_2^2(1+q)^2 - 2k^2\mu_2(1+q)\rho\sigma_1\sigma_2 + ((1+k)^2 - (1+2k)\rho^2)\sigma_1^2\sigma_2^2) \\ &= -4(1-\rho^2)\sigma_1^2\sigma_2^2((1+q)\mu_2 - \rho\sigma_1\sigma_2)^2 < 0 \\ &\implies k^2\mu_2^2(1+q)^2 - 2k^2\mu_2(1+q)\rho\sigma_1\sigma_2 + ((1+k)^2 - (1+2k)\rho^2)\sigma_1^2\sigma_2^2 > 0 \text{ for any } k. \end{aligned}$$

(End of the proof of Claim 5).

Finally, the definition of F and Claim 5 imply that $g'(x) = F(x, g(x))$ solves (5.7), and we complete the proof. \square

Proposition 5.3. *In case $0 < p < 1$ and $\mu_1 < \frac{\rho\mu_2\sigma_1}{\sigma_2}$, Proposition 4.1 holds.*

Figure 2. $0 < p < 1$ and $\mu_1 < \frac{\rho\mu_2\sigma_1}{\sigma_2}$ Figure 3. $p < 0$ and $\mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}$

Proof. By the same way as in Proposition 5.2, we can show that the level curve $C = 0$ is an ellipse, and the quadratic curves $A = 0$, $B = 0$, $C = 0$ are as in Figure 2. The region Ω is defined as

$$\Omega := \{(x, y) : x < 0, C(x, y) \geq 0, B(x, y) > 0, A(x, y) > 0\}.$$

As in Proposition 5.2, we can prove that there exist $\underline{x} < \bar{x} < 0$ and $g \in C^2([\underline{x}, \bar{x}])$ such that

$$g'(x) = F(x, g(x)), \quad g'(\underline{x}) = 0, \quad g'(\bar{x}) = 0, \quad \int_{\underline{x}}^{\bar{x}} \frac{g'(x)}{x} dx = \ln\left(\frac{1+\bar{\lambda}}{1-\underline{\lambda}}\right), \quad (5.8)$$

$$\text{where } F(x, y) := \frac{2C(x, y)}{-B(x, y) - \sqrt{B(x, y)^2 - 4A(x, y)C(x, y)}}.$$

Note that F is different from F in Proposition 5.2, but the analysis is almost same. Also we can prove the following properties of g by the same way as in Proposition 5.2:

- (i) $g(x) > 0$ and $g'(x) < 0$ for $x \in [\underline{x}, \bar{x}]$.
- (ii) $g'(x)/x > 0$ for $x \in (\underline{x}, \bar{x})$.
- (iii) $q g(x)(g'(x) + 1) - (1 + q)xg'(x) > 0$ for $x \in [\underline{x}, \bar{x}]$.
- (iv) $g(x) - xg'(x) > 0$ for $x \in [\underline{x}, \bar{x}]$.

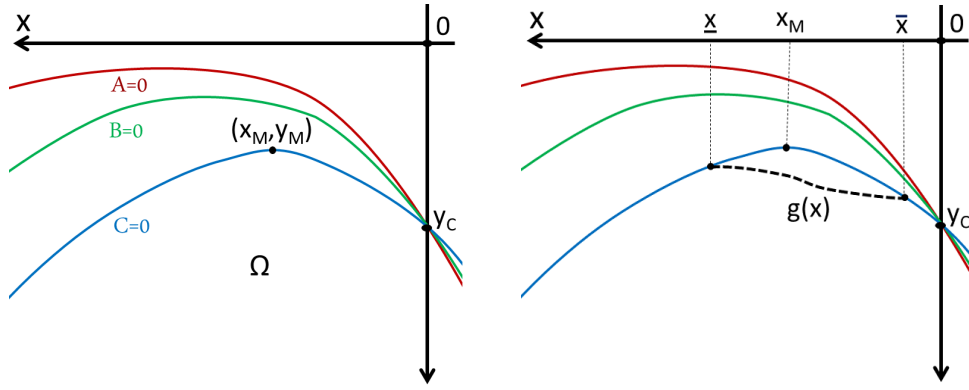
The proof is done by (5.8) and (i)-(iv). □

Proposition 5.4. *In case $p < 0$ and $\mu_1 > \frac{\rho\mu_2\sigma_1}{\sigma_2}$, Proposition 4.1 holds.*

Proof. Since $p < 0$, we have

$$q < 0, \quad 1 + q > 0, \quad 2\delta\sigma_2 - q\mu_2^2 > 0, \quad x_M > 0, \quad y_M < 0.$$

By the same way as in Proposition 5.2, we can show that the level curve $C = 0$ is a hyperbola, and the quadratic curves $A = 0$, $B = 0$, $C = 0$ are as in Figure 3 (we choose the lower curves of the

Figure 4. $p < 0$ and $\mu_1 < \frac{\rho\mu_2\sigma_1}{\sigma_2}$

hyperbola). Also,

$$\{(x, y) : x > 0, y < 0, C(x, y) \geq 0\} = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4,$$

$$\text{where } \begin{cases} \Omega_1 := \{(x, y) : x > 0, y < 0, C > 0, B \geq 0, A \geq 0\} \\ \Omega_2 := \{(x, y) : x > 0, y < 0, C > 0, B \geq 0, A < 0\} \\ \Omega_3 := \{(x, y) : x > 0, y < 0, C \geq 0, B < 0, A < 0\} \\ \Omega_4 := \{(x, y) : x > 0, y < 0, C \geq 0, B < 0, A \geq 0\} \end{cases}$$

Claim: For any $a \in \mathbb{R}$ such that $0 < a < x_M$, there exist a constant $b_a > a$ and a function $g_a : [a, b_a] \mapsto \mathbb{R}$ such that

$$g'_a(x) = F(x, g_a(x)), \quad g_a(a) = \Gamma(a), \quad g'_a(b_a) = 0,$$

$$\text{where } \begin{cases} F(x, y) := \frac{2C(x, y)}{-B(x, y) + \sqrt{B(x, y)^2 - 4A(x, y)C(x, y)}}, \\ \Gamma(x) := \frac{-q\sigma_2(\sigma_2 + (\mu_1\sigma_2 - \rho\sigma_1\mu_2)x) + \sigma_2\sqrt{2q\delta(\rho^2 - 1)\sigma_1^2\sigma_2^2x^2 + q^2(\mu_2^2\sigma_1^2x^2 - 2\rho\sigma_1\sigma_2\mu_2x(-1 + \mu_1x) + \sigma_2^2(-1 + \mu_1x)^2)}}{q(1 + q)(2\delta\sigma_2^2 - q\mu_2^2)}. \end{cases}$$

In fact, $y = \Gamma(x)$ is the equation of the lower curve of the hyperbola $C = 0$.

We observe that $\Gamma'(x) > 0$ for $0 < x < x_M$, $\Gamma'(x) < 0$ for $x > x_M$ and $\lim_{x \rightarrow \infty} \Gamma(x) = -\infty$. Using this observation, we can prove Claim by the same way as in Proposition 5.2.

Again, as in Proposition 5.2, there exist $0 < \underline{x} < \bar{x}$ and $g \in C^2([\underline{x}, \bar{x}])$ such that

$$g'(x) = F(x, g(x)), \quad g'(\underline{x}) = 0, \quad g'(\bar{x}) = 0, \quad \int_{\underline{x}}^{\bar{x}} \frac{g'(x)}{x} dx = \ln\left(\frac{1 + \bar{\lambda}}{1 - \underline{\lambda}}\right), \quad (5.9)$$

and g satisfies the following properties:

- (i) $g(x) < 0$ and $g'(x) > 0$ for $x \in [\underline{x}, \bar{x}]$.
- (ii) $g'(x)/x > 0$ for $x \in (\underline{x}, \bar{x})$.
- (iii) $qg(x)(g'(x) + 1) - (1 + q)xg'(x) > 0$ for $x \in [\underline{x}, \bar{x}]$.
- (iv) $g(x) - xg'(x) < 0$ for $x \in [\underline{x}, \bar{x}]$.

The proof is done by (5.9) and (i)-(iv). \square

Proposition 5.5. In case $p < 0$ and $\mu_1 < \frac{\rho\mu_2\sigma_1}{\sigma_2}$, Proposition 4.1 holds.

Proof. By the same way as in Proposition 5.4, we can show that the quadratic curves $A = 0$, $B = 0$, $C = 0$ are as in Figure 4. The region Ω is defined as

$$\Omega := \{(x, y) : x < 0, y < 0, C(x, y) \geq 0, B(x, y) > 0, A(x, y) > 0\}.$$

As in Proposition 5.4, we can prove that there exist $\underline{x} < \bar{x} < 0$ and $g \in C^2([\underline{x}, \bar{x}])$ such that

$$g'(x) = F(x, g(x)), \quad g'(\underline{x}) = 0, \quad g'(\bar{x}) = 0, \quad \int_{\underline{x}}^{\bar{x}} \frac{g'(x)}{x} dx = \ln\left(\frac{1+\bar{\lambda}}{1-\underline{\lambda}}\right), \quad (5.10)$$

where $F(x, y) := \frac{2C(x, y)}{-B(x, y) - \sqrt{B(x, y)^2 - 4A(x, y)C(x, y)}}$,

and g satisfies the following properties:

- (i) $g(x) < 0$ and $g'(x) < 0$ for $x \in [\underline{x}, \bar{x}]$.
- (ii) $g'(x)/x > 0$ for $x \in (\underline{x}, \bar{x})$.
- (iii) $qg(x)(g'(x) + 1) - (1 + q)xg'(x) > 0$ for $x \in [\underline{x}, \bar{x}]$.
- (iv) $g(x) - xg'(x) < 0$ for $x \in [\underline{x}, \bar{x}]$.

The proof is done by (5.10) and (i)-(iv). □

REFERENCES

- [1] M. AKIAN, J. MENALDI, AND A. SULEM, *On an investment-consumption model with transaction costs*, SIAM J. Control Optim., 34 (1996), pp. 329–364.
- [2] M. BICHUCH, *Asymptotic analysis for optimal investment in finite time with transaction costs*, SIAM Journal on Financial Mathematics, 3 (2012), pp. 433–458.
- [3] M. BICHUCH AND P. GUASONI, *Investing with liquid and illiquid assets*, <http://ssrn.com/abstract=2523538>, (2014). preprint.
- [4] M. BICHUCH AND S. E. SHREVE, *Utility maximization trading two futures with transaction costs*, SIAM Journal on Financial Mathematics, 4 (2013), pp. 26–85.
- [5] X. CHEN AND M. DAI, *Characterization of optimal strategy for multiasset investment and consumption with transaction costs*, SIAM J. Financial Math., 4 (2013), pp. 857–883.
- [6] J. CHOI, *Asymptotic analysis for Merton's problem with transaction costs in power utility case*, Stochastics, 86 (2014), pp. 803–816.
- [7] J. CHOI, M. SIRBU, AND G. ZITKOVIC, *Shadow prices and well-posedness in the problem of optimal investment and consumption with transaction costs.*, 2013, pp. 4414–4449.
- [8] M. DAI, H. JIN, AND H. LIU, *Illiquidity, position limits, and optimal investment for mutual funds*, J. Econom. Theory, 146 (2011), pp. 1598–1630.
- [9] M. H. A. DAVIS AND A. R. NORMAN, *Portfolio selection with transaction costs*, Math. Oper. Res., 15 (1990), pp. 676–713.
- [10] S. GERHOLD, P. GUASONI, J. MUHLE-KARBE, AND W. SCHACHERMAYER, *Transaction costs, trading volume, and the liquidity premium*, Finance Stoch., 18 (2014), pp. 1–37.
- [11] S. GERHOLD, J. MUHLE-KARBE, AND W. SCHACHERMAYER, *Asymptotics and duality for the Davis and Norman problem*, Stochastics, 84 (2012), pp. 625–641.
- [12] V. GINSBURGH AND M. KEYZER, *The structure of applied general equilibrium models*, vol. 2 of Applications of Mathematics (New York), The MIT Press, New York, 2002.
- [13] A. HERCZEGH AND V. PROKAJ, *Shadow price in the power utility case*, Ann. Appl. Probab., 25 (2015), pp. 2671–2707.
- [14] D. HOBSON, A. TSE, AND Y. ZHU, *A multi-asset investment and consumption problem with transaction costs*, <https://arxiv.org/abs/1612.01327>, (2016). preprint.
- [15] J. HUGONNIER AND D. KRAMKOV, *Optimal investment with random endowments in incomplete markets*, Annals of Applied Probability, (2004), pp. 845–864.
- [16] K. JANEČEK AND S. E. SHREVE, *Asymptotic analysis for optimal investment and consumption with transaction costs*, Finance Stoch., 8 (2004), pp. 181–206.
- [17] J. KALLSEN AND J. MUHLE-KARBE, *On using shadow prices in portfolio optimization with transaction costs*, Ann. Appl. Probab., 20 (2010), pp. 1341–1358.
- [18] I. KARATZAS, J. P. LEHOCZKY, AND S. E. SHREVE, *Optimal portfolio and consumption decisions for a “small investor” on a finite horizon*, SIAM J. Control Optim., 25 (1987), pp. 1557–1586.
- [19] I. KARATZAS, J. P. LEHOCZKY, S. E. SHREVE, AND G. L. XU, *Martingale and duality methods for utility maximization in an incomplete market*, SIAM J. Control Optim., 29 (1991), pp. 702–730.
- [20] I. KARATZAS AND S. E. SHREVE, *Methods of mathematical finance*, vol. 39 of Applications of Mathematics (New York), Springer-Verlag, New York, 1998.

- [21] I. KARATZAS AND G. ŽITKOVIĆ, *Optimal consumption from investment and random endowment in incomplete semimartingale markets*, The Annals of Probability, 31 (2003), pp. 1821–1858.
- [22] D. KRAMKOV AND W. SCHACHERMAYER, *The asymptotic elasticity of utility functions and optimal investment in incomplete markets*, Ann. Appl. Probab., 9 (1999), pp. 904–950.
- [23] H. LIU, *Optimal consumption and investment with transaction costs and multiple risky assets*, (2004), pp. 289–338.
- [24] M. J. P. MAGILL AND G. M. CONSTANTINIDES, *Portfolio selection with transactions costs*, J. Econom. Theory, 13 (1976), pp. 245–263.
- [25] R. C. MERTON, *Lifetime portfolio selection under uncertainty: The continuous-time case*, The review of Economics and Statistics, (1969), pp. 247–257.
- [26] ———, *Optimum consumption and portfolio rules in a continuous-time model*, J. Econom. Theory, 3 (1971), pp. 373–413.
- [27] K. MUTHURAMAN AND S. KUMAR, *Multidimensional portfolio optimization with proportional transaction costs*, Math. Finance, 16 (2006), pp. 301–335.
- [28] D. POSSAMAÏ, H. M. SONER, AND N. TOUZI, *Homogenization and Asymptotics for Small Transaction Costs: The Multidimensional Case*, Comm. Partial Differential Equations, 40 (2015), pp. 2005–2046.
- [29] S. E. SHREVE AND H. M. SONER, *Optimal investment and consumption with transaction costs*, Ann. Appl. Probab., 4 (1994), pp. 609–692.
- [30] A. V. SKOROHOD, *Stochastic equations for diffusion processes with a boundary*, Teor. Veroyatnost. i Primenen., 6 (1961), pp. 287–298.
- [31] W. WALTER, *Ordinary differential equations*, vol. 182 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1998. Translated from the sixth German (1996) edition by Russell Thompson, Readings in Mathematics.